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On Planar Curves with Position-Dependent Curvature

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Abstract
Motivated by homothetic solutions in curvature-driven flows of planar curves, as well as their many physical applications, this article carries out a systematic study of oriented smooth curves whose curvature κ is a given function of position or direction. The analysis is informed by a dynamical systems point of view. Though focussed on situations where the prescribed curvature depends only on the distance r from one distinguished point, the basic dynamical concepts are seen to be applicable in other situations as well. As an application, a complete classification of all closed solutions of $\kappa = ar^b$, with arbitrary real constants a, b, is established.

Keywords Curve shortening flow · Planar curve · Curvature · Net winding · Jordan solution

Mathematics Subject Classification 34C05 · 34C025 · 34C35 · 53A04 · 53C44 · 58F05 · 70H06

1 Introduction

Modulo rotations and translations, an oriented smooth planar curve is completely determined by its *curvature* [10, 17, 27]. Naturally, therefore, curvature plays a central role in the study of planar shapes. The evolution and characterization of planar shapes has been studied extensively and in a great variety of contexts, including curve flows [1, 4, 5, 8, 34], growth and abrasion processes [18, 19, 22, 24], optimization problems [20, 26], among many others. In these contexts, curvature typically is but one aspect of a more complicated process or model. Leaving aside all additional layers of complexity, the present article aims at classifying those planar curves whose curvature simply is a (given) function of position or direction. Formally, given any smooth function $k : U \times S^1 \to \mathbb{R}$, where $U \subset \mathbb{C}$ is open, it aims at studying all smooth curve solutions Z of

$$\kappa = k(Z, N); \tag{1.1}$$

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here $\kappa \in \mathbb{R}$ and $N \in S^1$ denote the curvature and unit normal vector at $Z(s) \in \mathbb{C}$, respectively. In particular, the article asks whether or not (1.1) allows for solutions that have further desirable properties such as being, for instance, simple, closed, or convex. The main goal is to address these questions in a systematic way, informed by a dynamical systems point of view. Although the pertinent dynamical ideas apply in greater generality, for concreteness most of the analysis is focussed on situations where k in (1.1) depends only on $|Z|$ and is independent of N, that is, on the special case

$$\kappa = f(|Z|), \tag{1.2}$$

where $f : \mathbb{R}^+ \to \mathbb{R}$ is a given smooth function. Thus, whereas solving (1.1) amounts to finding planar curves with prescribed position- and direction-dependent curvature, in (1.2) the prescribed curvature only depends on the distance from one distinguished point (namely, the origin).

One key tool in this article is a planar (topological) flow Φ_f associated with (1.2). Properties of Φ_f translate into properties of solutions of (1.2) that may be hard to recognize directly. For a simple illustration, take for instance $f(r) = r^4$. Figure 1 displays a few solutions of (1.2) in this case; see also the illustrations in [34]. As it turns out, the family of *all* (maximal) solutions of $\kappa = |Z|^4$ is most easily understood by considering the flow on $\mathbb{C}$ generated by

$$\dot{z} = iz|z|^4 - i. \tag{1.3}$$

Notice that (1.3) is Hamiltonian, with $H(z) = \mathsf{Re}\, z - \frac{1}{6}|z|^6$ being strictly convex and having a unique, non-degenerate global minimum at $z = 1$. Moreover, one of the main results of the present article, Theorem 5.6 below, implies that modulo rotations, $\kappa = |Z|^4$ has precisely *two* solutions that are simple closed (counter-clockwise oriented) curves: the unit circle and one non-circular oval; see Fig. 1.

One prominent natural source for (1.1) are curvature-driven curve flows which continue to be studied for their deep mathematical properties as well as their broad physical applications. Consider for example the *Andrews–Bloore flow* generated by

$$\frac{\partial Z_t}{\partial t} = -(a + \kappa^b)N\,; \tag{1.4}$$

here $(Z_t)_{0 \le t < T}$ with the appropriate $T > 0$ describes a parametrized family of oriented simple closed smooth curves, and $a, b \ge 0$ are real constants. Curve flows such as Andrews–Bloore have been suggested as simple models for a variety of physical processes, ranging from the growth of crystal surfaces to the abrasion of pebbles. As detailled in [22], some aspects of the dynamics of (1.4) in general remain a challenge, both mathematically and computationally. However, important special cases, including $b = 0$ (also referred to as the *eikonal flow*), $a = 0, b = 1$ (the *curve-shortening flow*), and $a = 0, b = \frac{1}{3}$ (the *affine curve-shortening flow*), are now well understood; see, e.g., [1, 4, 5, 34] and the many references therein. Since they may represent limiting or equilibrium shapes, *homothetic* solutions, i.e., solutions Z_t that are mere t-dependent rescalings of one fixed curve Z, play a key role in any analysis of (1.4). Assume for instance that $Z_t(s) = \varphi(t)Z(s)$, with $\varphi : [0, T[\to \mathbb{R}^+$, is a solution of (1.4) with $a = 0$. Then $\dot{\varphi} = \lambda\varphi^{-b}$ for some real constant $\lambda \neq 0$, and Z is a solution of

$$\kappa = \left|\lambda\, \mathsf{Re}\,(Z\overline{N})\right|^{1/b}. \tag{1.5}$$

While the (counter-clockwise oriented) circle with radius $|\lambda|^{-1/(1+b)}$ centered at the origin obviously solves (1.5), it is much less obvious whether or not any other (simple closed)

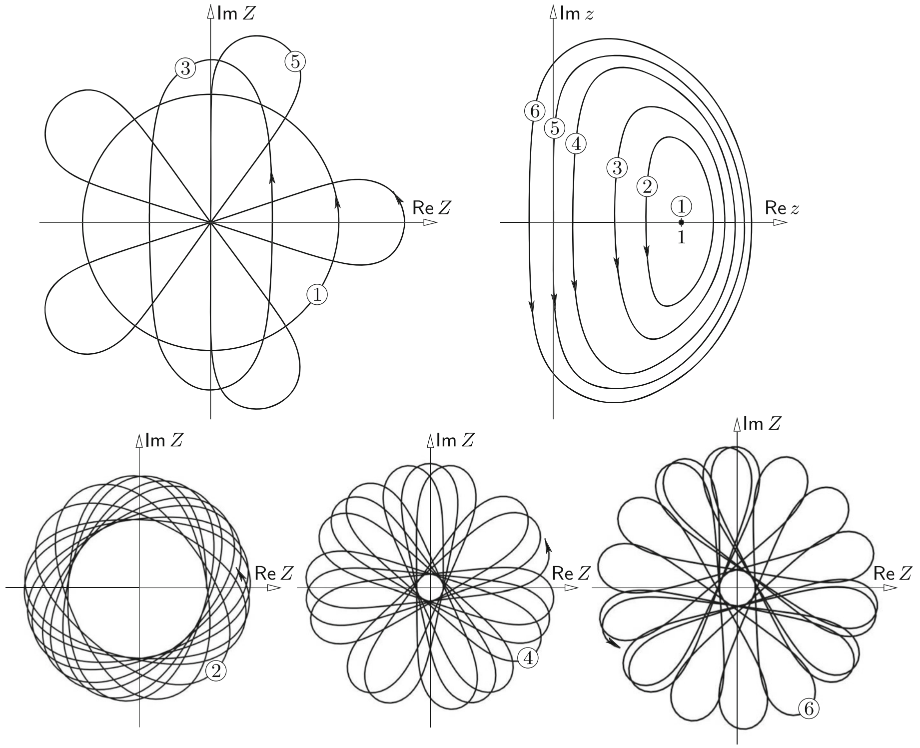

Fig. 1 Among all smooth-curve solutions of $\kappa = |Z|^4$, modulo rotations only precisely two are closed and simple: the unit circle and one non-circular oval, labelled 1 and 3, respectively (top left). The structure of *all* solutions is best understood by means of the associated planar flow generated by (1.3), notably its phase portrait (top right)

solutions exist. In essence, this question is answered in [5] by means of an ad-hoc analysis, with the answer depending on b in a non-trivial way. Note that (1.5) has the form (1.1). Towards the end of the present article, it will become clear that an analysis similar to, but simpler than the one presented here can be carried out, for instance, when k in (1.1) depends only on $\mathsf{Re}\,(Z\overline{N})$, a scenario that includes (1.5).

The unique non-circular oval solution of $\kappa = |Z|^4$ is readily seen to *not* be an ellipse. This observation nicely contrasts the ellipticity of all limiting shapes for the affine curve-shortening flow [5, 22]. From a physical point of view, therefore, if (1.2) were to be interpreted as describing (rescaled) limiting shapes of a curvature-driven abrasion process on pebbles, say, then such an interpretation, however physically questionable it may be in other respects, would at least be consistent with the well-documented empirical observation that shapes of worn stones are not exactly elliptical either, but rather appear to be a bit bulkier [18, 19, 23, 24].

Organization and notation

This article is organized as follows. Section 2 introduces the planar flow Φ_f associated with (1.2) and discusses a few of its basic properties. Section 3 establishes the crucial correspondance between the dynamics of Φ_f on the one hand and solutions of (1.2) on the other. For a reasonably wide class of smooth functions f, Sect. 4 characterizes simple closed solutions of (1.2) and provides tools to find all such solutions, or else to prove that none exist. Section 5 employs the machinery developed in earlier sections for an in-depth study of the monomial family $f(r) = r^b$, where b is an arbitrary real constant. A concluding supplemental section illustrates how the analysis, though quite specific and delicate, nonetheless is representative of arguments and techniques that can be applied to other classes of functions in (1.2), as well as variants thereof; as such, it is quite similar in spirit to analyses in, e.g., [5, 9, 29, 37].

The following, mostly standard symbols, notation, and terminology are used throughout. The sets of all positive integers, non-negative integers, integers, rational, positive real, real, and complex numbers are denoted $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}^+$, $\mathbb{R}$, and $\mathbb{C}$, respectively, and $\mathbb{R}_\infty := \mathbb{R} \cup \{-\infty, \infty\}$ is the extended real line with its familiar order, topology, and arithmetic [32, Sec. 1.22]. As usual, $\varnothing$ is the empty set, with $\inf \varnothing := \infty$ and $\sup \varnothing := -\infty$. Limits of real-valued objects (such as sequences, functions, or integrals) are understood in $\mathbb{R}_\infty$ unless stated otherwise. The terms *increasing* (respectively, *decreasing*) for sequences (a_n) in $\mathbb{R}_\infty$ are interpreted strictly, i.e., $a_n > a_m$ (respectively, $a_n < a_m$) whenever $n > m$, and similarly for functions. Usage of an inequality such as, e.g., $a > b$ with $a, b \in \mathbb{C}$ is understood to automatically imply that $a, b \in \mathbb{R}_\infty$. Numerical values of real numbers are displayed to four correct significant decimal digits. The real part, imaginary part, complex conjugate, and Euclidean norm of $z \in \mathbb{C}$ are $\mathsf{Re}\, z$, $\mathsf{Im}\, z$, $\overline{z}$, and $|z|$, respectively. For convenience, let $\mathbb{C}_\times := \mathbb{C} \setminus \{0\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and $S^1 = \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$. Also, for every $p, w \in \mathbb{C}$ and $\mathbb{A} \subset \mathbb{C}$ let $p + w\mathbb{A} = \{p + wz : z \in \mathbb{A}\}$, as well as $\overline{\mathbb{A}} = \{\overline{z} : z \in \mathbb{A}\}$ and $\mathrm{dist}(p, \mathbb{A}) = \inf_{z \in \mathbb{A}} |p - z|$, and denote the cardinality of $\mathbb{A}$ by $\#\mathbb{A}$. Moreover, $[p, w]$ is the closed line segment with end-points p, w, i.e., $[p, w] = \{(1-t)p + tw : 0 \le t \le 1\}$, and similarly for the open line segment $]p, w[$ etc. Given any function $f : \mathbb{A} \to \mathbb{R}_\infty$, write the set $\{z \in \mathbb{A} : f(z) = a\}$ simply as $\{f = a\}$, and its complement in $\mathbb{A}$ as $\{f \ne a\}$. As usual, Γ denotes the Euler Gamma function. In a slight abuse of familiar notation, let $\mathrm{O}(2)$ be the group of all isometries of $\mathbb{C}$ that fix 0. Recall that for every $Q \in \mathrm{O}(2)$ there exist $\vartheta \in \mathbb{R}$ and $\epsilon_Q \in \{-1, 1\}$ such that $Q(z) = e^{i\vartheta}(\mathsf{Re}\, z + i\epsilon_Q \mathsf{Im}\, z)$ for all $z \in \mathbb{C}$. Say that $\mathbb{A}, \mathbb{B} \subset \mathbb{C}$ are $\mathrm{O}(2)$-**congruent** if $Q(\mathbb{A}) = \mathbb{B}$ for some $Q \in \mathrm{O}(2)$.

2 An Auxiliary Planar Flow

Throughout, let $f : \mathbb{R}^+ \to \mathbb{R}$ be a smooth, that is, C^∞-function, with additional properties specified explicitly whenever needed. Given f, fix an $F : \mathbb{R}^+ \to \mathbb{R}$ with $F'(s) = sf(s) - 1$ for all $s \in \mathbb{R}^+$. (The particular choice of F is not going to matter prior to Proposition 4.7 below.) Also, let $\mathbb{F}_f = \big\{z \in \mathbb{C}_\times : zf(|z|) = 1\big\}$, a (possibly empty) subset of the real axis. On $\mathbb{C}_\times$, consider the ODE for $z = z(t)$,

$$\dot{z} = izf(|z|) - i. \tag{2.1}$$

Recall that (2.1) generates a local flow Φ_f on $\mathbb{C}_\times$, in that for every $p \in \mathbb{C}_\times$ there exists an open interval $\mathbb{J} \subset \mathbb{R}$ with $0 \in \mathbb{J}$ such that $z(t) = \Phi_f(t, p)$ for all $t \in \mathbb{J}$ yields the unique, non-extendable solution of (2.1) with $z(0) = p$; see, e.g., [7, 10, 30]. Since $\big|\mathrm{d}|z|/\mathrm{d}t\big| = \big| - \mathsf{Im}\, z/|z|\big| \le 1$, it is impossible that $z(t) \to \infty$ for finite t. By contrast, it is possible that $z(t) \to 0$ for finite t; as recorded in Proposition 2.1 below, however, such behaviour is unproblematic. More precisely, Φ_f can be extended to a unique (global) flow on $\mathbb{C}$, henceforth also denoted Φ_f. (Here and throughout, the term *flow* is understood to mean *topological flow*, so Φ_f corresponds to a one-parameter group of homeomorphisms of $\mathbb{C}$; see [25, Sec. 1.I] and Remark 2.3(v) below.) If $\limsup_{s\to 0}|f'(s)| < \infty$ then this immediately follows from standard facts [3, 7, 30, 36], but otherwise a more tailor-made argument is required. With a view towards the subsequent analysis of (2.1), one such argument utilizes the smooth function $H_f : \mathbb{C}_\times \to \mathbb{R}$ given by

$$H_f(z) = \mathsf{Re}\, z - F(|z|) - |z| \quad \forall z \in \mathbb{C}_\times .$$

Evidently, $H = H_f$ has the property that $H(z) - \mathsf{Re}\, z$ is constant along circles centered at 0, i.e.,

$$\text{for every } s \in \mathbb{R}^+ \text{ there exists } a \in \mathbb{R} \text{ such that } H(z) - \mathsf{Re}\, z = a \quad \text{for all } |z| = s. \tag{2.2}$$

Every smooth function $H : \mathbb{C}_\times \to \mathbb{R}$ satisfying (2.2) equals H_f for an appropriate f — simply take $f(s) = (1 - \mathrm{d}H(s)/\mathrm{d}s)/s$. Most importantly, H_f is a first integral of (2.1), and hence the study of the latter ODE for the most part reduces to an analysis of the level sets of H_f. Notice that $H_f(\overline{z}) = H_f(z)$ for all $z \in \mathbb{C}_\times$, so all level sets are symmetric w.r.t. the real axis. Correspondingly, (2.1) has a basic symmetry as well: If $z(\cdot)$ is a solution then so is its **conjugate-reversal** $\overline{z(-\,\cdot)}$. Utilizing H_f, it is a routine exercise to establish the basic fact alluded to earlier.

Proposition 2.1 *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be smooth. Then (2.1) generates a unique flow Φ_f on $\mathbb{C}$, and*

$$\overline{\Phi_f(t, z)} = \Phi_f(-t, \overline{z}) \quad \forall (t, z) \in \mathbb{R} \times \mathbb{C}. \tag{2.3}$$

For the flow Φ_f, the time-t map $\Phi_f(t, \cdot)$ is a homeomorphism of $\mathbb{C}$ for every $t \in \mathbb{R}$, and $\Phi_f(\mathbb{R}, z) := \{\Phi_f(t, z) : t \in \mathbb{R}\}$ is the **orbit** of $z \in \mathbb{C}$. To exploit the basic symmetry (2.3), say that $z, p \in \mathbb{C}$ are Φ_f-**conjugate** if $\Phi_f(t, z) \in \{p, \overline{p}\}$ for some $t \in \mathbb{R}$. Plainly, Φ_f-conjugacy is an equivalence relation, and z, p are Φ_f-conjugate if and only if $\Phi_f(\mathbb{R}, z)$ equals either $\Phi_f(\mathbb{R}, p)$ or $\overline{\Phi_f(\mathbb{R}, p)}$. For every $z \in \mathbb{C}$, let $T_f(z) = \inf\{t \in \mathbb{R}^+ : \Phi_f(t, z) = z\}$. Thus $z \in \mathbb{C}$ is a periodic (respectively, fixed) point of Φ_f, or $\Phi_f(\mathbb{R}, z)$ is a periodic orbit, in symbols $z \in \mathrm{Per}\,\Phi_f$ (respectively, $z \in \mathrm{Fix}\,\Phi_f$), if and only if $T_f(z) < \infty$ (respectively, $T_f(z) = 0$). Notice that if $z \in \mathrm{Per}\,\Phi_f$ then z, w are Φ_f-conjugate precisely if $\Phi_f(t, z) = w$ for some $0 \le t < T_f(z)$. Also, $\mathrm{Fix}\,\Phi_f \setminus \{0\} = \mathbb{F}_f$, whereas 0 may or may not be a fixed point; see Examples 2.10 to 2.12 below. To clarify the nature of the fixed points of Φ_f, recall the notions of a (topological) saddle and a center, e.g., from [30, Sec. 2.10]. Specifically, $z \in \mathbb{C}$ is a **center** of Φ_f if a punctured neighbourhood of z is the disjoint union of periodic orbits, each of which has z in its interior. For instance, the linearization of (2.1) at $z \in \mathbb{F}_f$ is

$$\dot{p} = iF''(|z|)\,\mathsf{Re}\, p - f(|z|)\,\mathsf{Im}\, p, \tag{2.4}$$

suggesting that z is a center when $zF''(|z|) > 0$, and a saddle when $zF''(|z|) < 0$; see Lemma 2.5 below. The origin may be a fixed point as well; if it is, it cannot be a saddle due to (2.2), but may be a center, a situation that is straightforward to characterize.

Proposition 2.2 *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be smooth. Then 0 is a center of Φ_f if and only if both of the following conditions hold:*

(i) $\mathrm{dist}(0, \mathbb{F}_f) > 0$*;*
(ii) for every $a \in \mathbb{R}$ there exists a decreasing sequence (s_n) with $\lim_{n\to\infty} s_n = 0$ such that $|F(s_n) + s_n + a| > s_n$ for all n.

Remark 2.3 (i) By Proposition 2.2, the point 0 is a center of Φ_f precisely if it is not an accumulation point of $\mathbb{F}_f$, and $F(0+) := \lim_{s\to 0} F(s)$ either does not exist in $\mathbb{R}$, or else $\big(F(s) - F(0+)\big)/s \notin [-2, 0]$ for arbitrarily small $s \in \mathbb{R}^+$. Letting $a = \liminf_{s\to 0} s|f(s)|$, therefore, 0 is a center when $a > 1$, but is not a center when $a < 1$; if $a = 1$ then 0 may or may not be a center as the examples $f(s) = 1 + 1/s$ and $f(s) = 1/s$ show.

(ii) Though this is of no direct consequence for the present article, note that (2.1) actually is Hamiltonian since

$$\frac{\mathrm{dRe}\, z}{\mathrm{d}t} = \frac{\partial H_f}{\partial \mathrm{Im}\, z}, \quad \frac{\mathrm{dIm}\, z}{\mathrm{d}t} = -\frac{\partial H_f}{\partial \mathrm{Re}\, z}.$$

Being a 1-DOF Hamiltonian flow severely constrains the dynamical complexity of Φ_f, notably the nature of its fixed and periodic points (e.g., no sources, sinks, or limit cycles).

(iii) In the setting of Proposition 2.1, note that

$$\Phi_{-f}(t, z) = -\Phi_f(-t, -z) \quad \forall (t, z) \in \mathbb{R} \times \mathbb{C}.$$

As far as the dynamics of Φ_f is concerned, therefore, f may be replaced by $-f$ whenever convenient, e.g., if $f(s) \neq 0$ or $f'(s) \neq 0$ for all $s \in \mathbb{R}^+$ then it may be assumed that $f > 0$ or that f is increasing, respectively.

(iv) With $\Phi_f(t, \infty) := \infty$ for all $t \in \mathbb{R}$, the flow Φ_f may be considered a flow on the compactified complex plane $\mathbb{C} \cup \{\infty\}$. In this setting, the fixed point ∞ cannot be a saddle, but may be a centre. In perfect analogy to Proposition 2.2, it is straightforward to show that ∞ is a center of Φ_f if and only if $\mathbb{F}_f$ is bounded, and for every $a \in \mathbb{R}$ there exists an increasing sequence (s_n) with $\lim_{n\to\infty} s_n = \infty$ such that $|F(s_n) + s_n + a| > s_n$ for all n; see Proposition 2.7 below.

(v) Smoothness of $f : \mathbb{R}^+ \to \mathbb{R}$ is assumed throughout for convenience only. All results remain valid under appropriate finite differentiability assumptions; in most instances it suffices to assume the function f to be C^1. Also note that Φ_f is not in general a smooth flow, due to $f(|z|)$ being non-smooth or indeed undefined at $z = 0$. However, if for instance f is an even polynomial, as it is, e.g., in (1.3), then clearly Φ_f is smooth (on $\mathbb{R} \times \mathbb{C}$).

Given $z \in \mathrm{Per}\,\Phi_f$, say that the orbit $\Phi_f(\mathbb{R}, z)$ is **untwisted** if 0 lies in the exterior of the closed path $\Phi_f(\cdot, z)$; otherwise $\Phi_f(\mathbb{R}, z)$ is **twisted**. Note that $\Phi_f(\mathbb{R}, 0)$, if at all periodic, is twisted. By contrast, $\Phi_f(\mathbb{R}, z) = \{z\}$ is untwisted for every $z \in \mathbb{F}_f$. Given $z \in \mathrm{Per}\,\Phi_f \setminus \Phi_f(\mathbb{R}, 0)$, define the **net winding** of z as

$$\omega_f(z) = \pm\frac{1}{2\pi}\int_0^{T_f(z)} f(|\Phi_f(t, z)|)\,\mathrm{d}t, \tag{2.5}$$

where the plus sign (respectively, minus sign) applies in (2.5) when $\Phi_f(\cdot, z)$ is oriented counter-clockwise (respectively, clockwise). Net winding plays a key role in later sections. Here only a few basic properties are recorded. Clearly, ω_f is constant along orbits. If $z \in \Phi_f(\mathbb{R}, 0)$ then the integral in (2.5) may or may not exist (in $\mathbb{R}_\infty$); it does exist, for instance, if $f \geq 0$ or $f' \geq 0$. Notice, however, that strictly speaking $\omega_f(z)$ is defined only for $z \in \mathrm{Per}\,\Phi_f \setminus \Phi_f(\mathbb{R}, 0)$. The following is an immediate consequence of (2.1) and (2.5).

Proposition 2.4 *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be smooth, and $z \in \mathrm{Per}\,\Phi_f \setminus \Phi_f(\mathbb{R}, 0)$. Then, with the same signs as in (2.5),*

$$\omega_f(z) - k_z = \pm\frac{1}{2\pi}\int_0^{T_f(z)} \frac{\mathrm{d}t}{\Phi_f(t, z)} = \pm\frac{1}{2\pi}\int_0^{T_f(z)} \frac{\mathrm{Re}\,\Phi_f(t, z)}{|\Phi_f(t, z)|^2}\,\mathrm{d}t, \tag{2.6}$$

where $k_z = 0$ or $k_z = 1$ when $\Phi_f(\mathbb{R}, z)$ is untwisted or twisted, respectively.

On the (possibly empty or disconnected) set $\text{Per}\,\Phi_f \setminus (\Phi_f(\mathbb{R}, 0) \cup \text{Fix}\,\Phi_f)$, the function ω_f is continuous, but it is not in general continuous at $z \in \text{Fix}\,\Phi_f$ as, for instance, $\omega_f(z) = 0 \neq \lim_{p\to z}\omega_f(p)$ for $z \in \mathbb{F}_f$, provided that z is a non-degenerate center. (Recall that $\mathbb{F}_f \subset \mathbb{R}$.)

Lemma 2.5 *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be smooth. If $z \in \mathbb{F}_f$ and $zF''(|z|) > 0$ then z is a center of Φ_f, and*

$$\lim_{p\to z}\omega_f(p) = \frac{1}{\sqrt{zF''(|z|)}}. \tag{2.7}$$

Proof It is readily seen that $z \in \mathbb{F}_f$ is a non-degenerate maximum or minimum of H_f if and only if $f(|z|)F''(|z|) = zF''(|z|)/|z|^2 > 0$. In this case, z is a center, and $f(|\Phi_f(t, p)|) \to f(|z|)$ uniformly in t as $p \to z$, whereas $T_f(p) \to 2\pi/\sqrt{f(|z|)F''(|z|)}$, the minimal period of (2.4). Consequently,

$$\lim_{p\to z}\omega_f(p) = \pm\frac{1}{2\pi}f(|z|)\cdot\frac{2\pi}{\sqrt{f(|z|)F''(|z|)}} = \pm\frac{|z|}{z\sqrt{zF''(|z|)}},$$

and since $\Phi_f(\cdot, z)$ is oriented counter-clockwise (respectively, clockwise) when $z > 0$ (respectively, $z < 0$), this proves (2.7). □

Recall from Remark 2.3(i) that 0 is a center of Φ_f whenever $\liminf_{s\to 0} s|f(s)| > 1$. Under a slightly stronger assumption, the behaviour of ω_f near 0 is as follows.

Lemma 2.6 *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be smooth. If $\lim_{s\to 0} s|f(s)| = a > 1$ then 0 is a center of Φ_f, and*

$$\lim_{z\to 0}\omega_f(z) = \frac{1}{\sqrt{1 - 1/a^2}}. \tag{2.8}$$

Proof By Proposition 2.2, 0 is a center, and clearly $s_0\mathbb{D} \setminus \{0\} \subset \text{Per}\,\Phi_f \setminus \text{Fix}\,\Phi_f$ for some $s_0 \in \mathbb{R}^+$. To establish (2.8), assume first that $a = \infty$, and in fact $\lim_{s\to\infty} sf(s) = \infty$. Given any $b \in \mathbb{R}^+$, it can be assumed that $|\Phi_f(t, z)|f(|\Phi_f(t, z)|) \geq b + 1$ for all $0 < |z| < s_0$ and all $t \in \mathbb{R}$. Noting that $\Phi_f(\cdot, z)$ winds around 0 counter-clockwise, write $\Phi_f(t, z) = \rho e^{i\varphi}$, with smooth functions $\rho = \rho(t) > 0$ and $\varphi = \varphi(t)$. With this, (2.1) reads

$$\dot\rho = -\sin\varphi, \quad \dot\varphi = f(\rho) - \frac{\cos\varphi}{\rho}.$$

Note that $\dot\varphi \geq (\rho f(\rho) - 1)/\rho \geq b/\rho > 0$. It follows that

$$\omega_f(z) = \frac{1}{2\pi}\int_0^{T_f(z)} f(\rho)\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{T_f(z)}\left(\dot\varphi + \frac{\cos\varphi}{\rho}\right)\mathrm{d}t = 1 + \frac{1}{2\pi}\int_0^{T_f(z)}\frac{\cos\varphi}{\rho}\,\mathrm{d}t,$$

and consequently

$$|\omega_f(z) - 1| \leq \frac{1}{2\pi}\int_0^{T_f(z)}\frac{\mathrm{d}t}{\rho} \leq \frac{1}{2\pi}\int_0^{T_f(z)}\frac{\dot\varphi}{b}\,\mathrm{d}t = \frac{1}{b}.$$

Since $b \in \mathbb{R}^+$ has been arbitrary, $\lim_{z\to 0}\omega_f(z) = 1$. The argument in case $\lim_{s\to 0} sf(s) = -\infty$ is completely analogous, and hence (2.8) is correct when $a = \infty$.

It remains to consider the case $1 < a < \infty$. Assume first that $\lim_{s\to 0} sf(s) = a$. Note that $\lim_{s\to 0} F(s)$ exists in $\mathbb{R}$, and so does $\lim_{z\to 0} H_f(z)$. Since 0 is a center, it suffices to consider $\omega_f(s)$ for sufficiently small $s \in \mathbb{R}^+$. For every such s, there exists a unique $0 < s^* < s$

with $H_f(-s^*) = H_f(s) = -F(s)$, or equivalently $s^* + s = \int_{s^*}^{s} uf(u)\,\mathrm{d}u$. From the latter, it is easily deduced that $\lim_{s\to 0} s^*/s = (a-1)/(a+1)$. By the symmetry of (2.1) and $\mathrm{d}|z|/\mathrm{d}t = -\mathsf{Im}\, z/|z|$,

$$\omega_f(s) = \frac{1}{\pi}\int_0^{\frac{1}{2}T_f(s)} f(|\Phi_f(t,s)|)\,\mathrm{d}t = \frac{1}{\pi}\int_{s^*}^{s} \frac{uf(u)}{y(u)}\,\mathrm{d}u,$$

where $y = y(u) \ge 0$ is determined uniquely by $|x+iy| = u$ and $H_f(x+iy) = -F(s)$, that is,

$$\begin{aligned} y^2 = u^2 - x^2 = u^2 - \big(u + F(u) - F(s)\big)^2 &= \big(F(s) - F(u)\big)\big(2u + F(u) - F(s)\big) \\ &= (s-u)\left(\frac{1}{s-u}\int_u^s vf(v)\,\mathrm{d}v - 1\right)(u - s^*)\left(\frac{1}{u-s^*}\int_{s^*}^u vf(v)\,\mathrm{d}v + 1\right), \end{aligned}$$

and consequently

$$\omega_f(s) = \frac{1}{\pi}\int_{s^*/s}^{1} \frac{\widehat{f_s}(u)\,\mathrm{d}u}{\sqrt{(1-u)(u-s^*/s)}},$$

where $\widehat{f_s} : [0,1] \to \mathbb{R}$ is the continuous function with

$$\widehat{f_s}(u) = \frac{suf(su)}{\sqrt{\frac{1}{1-u}\int_u^1 svf(sv)\,\mathrm{d}v - 1}\sqrt{\frac{1}{u-s^*/s}\int_{s^*/s}^u svf(sv)\,\mathrm{d}v + 1}} \quad \forall 0 < u < 1, u \ne \frac{s^*}{s}.$$

Notice that $\lim_{s\to 0} \widehat{f_s}(u) = a/\sqrt{a^2-1}$ uniformly on [0, 1], and hence

$$\lim\nolimits_{s\to 0} \omega_f(s) = \frac{1}{\pi}\int_{\frac{a-1}{a+1}}^{1} \frac{a}{\sqrt{a^2-1}} \cdot \frac{\mathrm{d}u}{\sqrt{(1-u)\big(u - \frac{a-1}{a+1}\big)}} = \frac{1}{\sqrt{1-1/a^2}}.$$

This establishes (2.8) when $\lim_{s\to 0} sf(s) = a$, and again the case $\lim_{s\to 0} sf(s) = -a$ is completely analogous. □

In order for $\omega_f(z)$ to be defined whenever $|z|$ is large, note that $z \in \mathrm{Per}\,\Phi_f \setminus \mathrm{Fix}\,\Phi_f$ for all sufficiently large $|z|$, provided that $\liminf_{s\to\infty} s|f(s)| > 1$. In the terminology of Remark 2.3(iv), the fixed point ∞ is a center of Φ_f in this case. The following, then, is an analogue of Lemma 2.6; its very similar proof is left to the interested reader.

Proposition 2.7 *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be smooth. If $\lim_{s\to\infty} s|f(s)| = a > 1$ then every solution of (2.1) is bounded, $\mathbb{C} \setminus s\mathbb{D} \subset \mathrm{Per}\,\Phi_f \setminus \mathrm{Fix}\,\Phi_f$ for some $s \in \mathbb{R}^+$, and*

$$\lim\nolimits_{z\to\infty} \omega_f(z) = \frac{1}{\sqrt{1-1/a^2}}.$$

Remark 2.8 By its very definition (2.5), the function ω_f bears some resemblance to the minimal period function T_f. The literature on minimal periods in Hamiltonian systems, notably near non-degenerate centers, is substantial; see, e.g., [9, 11–15, 31, 35, 36, 38] and the many references therein. The author does not know whether these fine studies can fruitfully be applied for the purpose of the present article, and in particular whether a multiple of ω_f can be interpreted as the *true* period in a 1-DOF Hamiltonian flow. Usage of ω_f in later sections may also remind the reader of the basic differential geometry notions of total curvature and rotation index [10, 17, 27, 28]. Unlike the latter, however, the value of ω_f need not be an integer but can in fact be any (extended) real number.

For the analysis in later sections, it is crucial whether or not ω_f attains certain particular values. To state a simple first observation in this regard, note that if $f(s) \neq 0$ or $f'(s) \neq 0$ for all $s \in \mathbb{R}^+$ then $\omega_f(z)$ is well-defined (in $\mathbb{R}_\infty$) for every $z \in \text{Per}\,\Phi_f$, unless $z = 0 \in \text{Fix}\,\Phi_f$, in which case simply define $\omega_f(0) = 0$. With this, $\omega_f(z) = 0$ for every $z \in \text{Fix}\,\Phi_f$. Moreover, the possible values of ω_f always are constrained as follows.

Lemma 2.9 *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be smooth, and $z \in \text{Per}\,\Phi_f \setminus \text{Fix}\,\Phi_f$.*

(i) If $f(s) \neq 0$ for all $s \in \mathbb{R}^+$ then $\omega_f(z) > 0$.
(ii) If $f'(s) \neq 0$ for all $s \in \mathbb{R}^+$ then $\omega_f(z) \neq 1$.
(iii) If $f(s)f'(s) \neq 0$ for all $s \in \mathbb{R}^+$ then $0 < \omega_f(z) < 1$ when $ff' > 0$, and $\omega_f(z) > 1$ when $ff' < 0$.

Proof To see (i), simply note that $\Phi_f(\cdot, z)$ is positively (respectively, negatively) oriented when $f > 0$ (respectively, $f < 0$), and hence $\omega_f(z) > 0$ in either case, by (2.5).

To prove (ii), let $s_1 < s_2$ be the intersection points of $\Phi_f(\mathbb{R}, z)$ with the real axis. For convenience, let $a = H_f(z) = H_f(s_1) = H_f(s_2)$. By Remark 2.3(iii), it may be assumed that $f' > 0$. Let $s_0 = \inf\{f > 0\}$. If $\max\{|s_1|, |s_2|\} \le s_0$ then, utilizing (2.1) and its symmetry,

$$\omega_f(z) = \omega_f(s_1) = -\frac{1}{\pi}\int_0^{\frac{1}{2}T_f(s_1)} f(|\Phi_f(t, s_1)|)\,\mathrm{d}t = \frac{1}{\pi}\int_{s_1}^{s_2} \frac{\mathrm{d}s}{y(s)},$$

where $y = y(s) > 0$ for $s_1 < s < s_2$ is given implicitly by $H_f(s + iy) = a$. Since f is increasing, it is readily seen that the relevant component of the level set $\{H_f = a\}$ intersects the set $\left\{z \in \mathbb{C} : (s_1 + s_2)\mathsf{Re}\, z = |z|^2 + s_1 s_2\right\}$, i.e., the circle with radius $\frac{1}{2}(s_2 - s_1)$ centered at $\frac{1}{2}(s_2 + s_1)$, only in s_1, s_2, or, more algebraically, $y(s) \neq \sqrt{(s_2 - s)(s - s_1)}$ for all $s_1 < s < s_2$. With this,

$$\begin{aligned}\omega_f(z) - 1 &= \tfrac{1}{\pi}\int_{s_1}^{s_2} \tfrac{\mathrm{d}s}{y(s)} - \tfrac{1}{\pi}\int_{s_1}^{s_2} \tfrac{\mathrm{d}s}{\sqrt{(s_2-s)(s-s_1)}} \\ &= \tfrac{1}{\pi}\int_{s_1}^{s_2} \tfrac{\sqrt{(s_2-s)(s-s_1)} - y(s)}{y(s)\sqrt{(s_2-s)(s-s_1)}}\,\mathrm{d}s \neq 0.\end{aligned} \tag{2.9}$$

Since a virtually identical argument applies when $\min\{|s_1|, |s_2|\} \ge s_0$, it only remains to consider the case $\min\{|s_1|, |s_2|\} < s_0 < \max\{|s_1|, |s_2|\}$. Thus assume for instance that $|s_1| > s_0 > |s_2|$. There exists a unique $s_2 < s_3 < s_0$ such that $\mathsf{Re}\,\dot{z} = 0$ when $\mathsf{Re}\, z = s_3$, and consequently

$$\begin{aligned}\omega_f(z) &= -\frac{1}{\pi}\int_{s_2}^{s_3} \frac{\mathrm{d}s}{y^-(s)} - \frac{1}{\pi}\int_{s_3}^{s_1} \frac{\mathrm{d}s}{y^+(s)} \\ &= \frac{1}{\pi}\int_{s_1}^{s_2} \frac{\mathrm{d}s}{y^+(s)} + \frac{1}{\pi}\int_{s_2}^{s_3}\left(\frac{1}{y^+(s)} - \frac{1}{y^-(s)}\right)\mathrm{d}s < \frac{1}{\pi}\int_{s_1}^{s_2} \frac{\mathrm{d}s}{y^+(s)},\end{aligned}$$

where $0 < y^-(s) < y^+(s)$ for $s_2 < s < s_3$ are the two solutions of $H_f(s + iy) = a$. Since $y^+ > \sqrt{(s_2 - s)(s - s_1)}$ for all $s_1 < s < s_2$, similarly to (2.9),

$$\omega_f(z) - 1 < \frac{1}{\pi}\int_{s_1}^{s_2} \frac{\mathrm{d}s}{y^+(s)} - 1 = \frac{1}{\pi}\int_{s_1}^{s_2} \frac{\sqrt{(s_2 - s)(s - s_1)} - y^+(s)}{y^+(s)\sqrt{(s_2 - s)(s - s_1)}}\,\mathrm{d}s < 0.$$

A completely analogous argument applies when $|s_1| < s_0 < |s_2|$.

To prove (iii), again assume w.l.o.g. that $f' > 0$. Using the same quantities as in the proof of (ii), it is readily checked that $f > 0$ implies $y(s) > \sqrt{(s_2 - s)(s - s_1)}$ for all $s_1 < s < s_2$, whereas this inequality is reversed when $f < 0$. By (2.9), therefore, $\omega_f(z) < 1$ when $f > 0$, and $\omega_f(z) > 1$ when $f < 0$. □

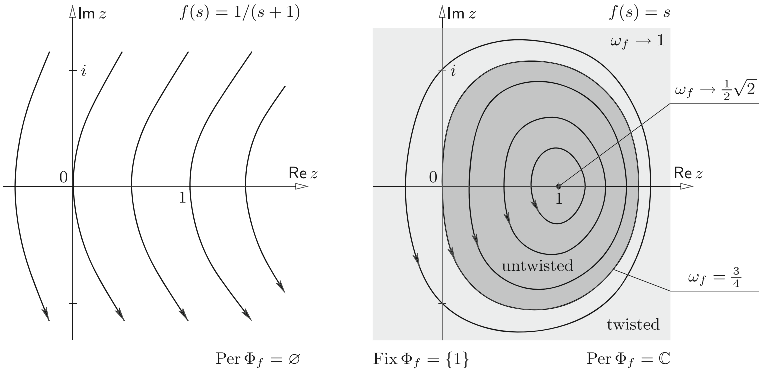

Fig. 2 For $f(s) = 1/(1+s)$, no point is periodic under Φ_f (left; see Example 2.10), whereas for $f(s) = s$ every point is periodic, with the center 1 being the only fixed point (right; see Example 2.11)

The following examples illustrate the notions introduced in this section; in particular, they show how 0 may be non-periodic, a periodic but non-fixed point, or a fixed point of Φ_f, respectively.

Example 2.10 Let $f(s) = 1/(1+s)$ for all $s \in \mathbb{R}^+$. Then Per $\Phi_f = \varnothing$, and every solution of (2.1) is unbounded. In particular, 0 is non-periodic, its orbit given implicitly by $\mathsf{Re}\, z = |z| - \log(1+|z|)$; see Fig. 2.

Example 2.11 Let $f(s) = s$ for all $s \in \mathbb{R}^+$. Then Fix $\Phi_f = \mathbb{F}_f = \{1\}$, and 1 is a center with $F''(1) = 2$, so $\lim_{z\to 1} \omega_f(z) = \frac{1}{2}\sqrt{2}$. Every orbit is periodic, and correspondingly every solution of (2.1) is bounded. In particular, 0 is periodic with $T_f(0) = \frac{1}{4}\sqrt{6}\,\Gamma(\frac{1}{4})^2/\sqrt{\pi} = 4.541$, its orbit given implicitly by $3\,\mathsf{Re}\, z = |z|^3$, and $\omega_f(0) = \frac{3}{4}$. Proposition 2.7 and Lemma 2.9 imply that $\lim_{z\to\infty} \omega_f(z) = 1$, and $0 < \omega_f(z) < 1$ for every $z \in \mathbb{C} \setminus \{1\}$; see Fig. 2.

Example 2.12 This example illustrates several different ways how 0 may be a fixed point of Φ_f; see Fig. 3.

(i) Let $f(s) = 1/s^2$ for all $s \in \mathbb{R}^+$. Then $\mathbb{F}_f = \{1\}$, and $F''(1) = -1$, so the fixed point 1 is a saddle. By Proposition 2.2, the fixed point 0 is a center, and $\lim_{z\to 0} \omega_f(z) = 1$, by Lemma 2.6. All other periodic orbits are twisted; they lie inside the homoclinic loop associated with the saddle and given implicitly by $\mathsf{Re}\, z = 1 + \log|z|$. By Lemma 2.9, $\omega_f(z) > 1$ for every $z \in$ Per $\Phi_f \setminus \{0, 1\}$, and $\omega_f(z) \to \infty$ as z approaches the homoclinic loop.

(ii) Let $f(s) = 1/s$ for all $s \in \mathbb{R}^+$. Then Fix Φ_f = Per $\Phi_f = \{z \in \mathbb{C} : \mathsf{Re}\, z \geq 0, \mathsf{Im}\, z = 0\}$, hence the fixed point 0 is not isolated. Every point not on the non-negative real axis has an unbounded orbit which in fact is a parabola.

(iii) Let $f(s) = (1+3s^4)/(s+3s^3)$ for all $s \in \mathbb{R}^+$. Again, $\mathbb{F}_f = \{1\}$, but $F''(1) = \frac{3}{2}$, so unlike in (i), the fixed point 1 now is a center. Every point not on the homoclinic loop associated with the fixed point 0 and given implicitly by $3\,\mathsf{Re}\, z = |z|^3 - |z| + \frac{4}{3}\sqrt{3}\arctan(|z|\sqrt{3})$ is periodic. Note that $\omega_f(z) \to \infty$ as z approaches the homoclinic loop. Also, $\lim_{z\to 1} \omega_f(z) = \frac{1}{3}\sqrt{6} < 1$ and $\lim_{z\to\infty} \omega_f(z) = 1$. Since $f'(0.8294) = 0$, Lemma 2.9(ii,iii) do not apply. By the intermediate value theorem there exists at least one untwisted periodic orbit for which $\omega_f = 1$.

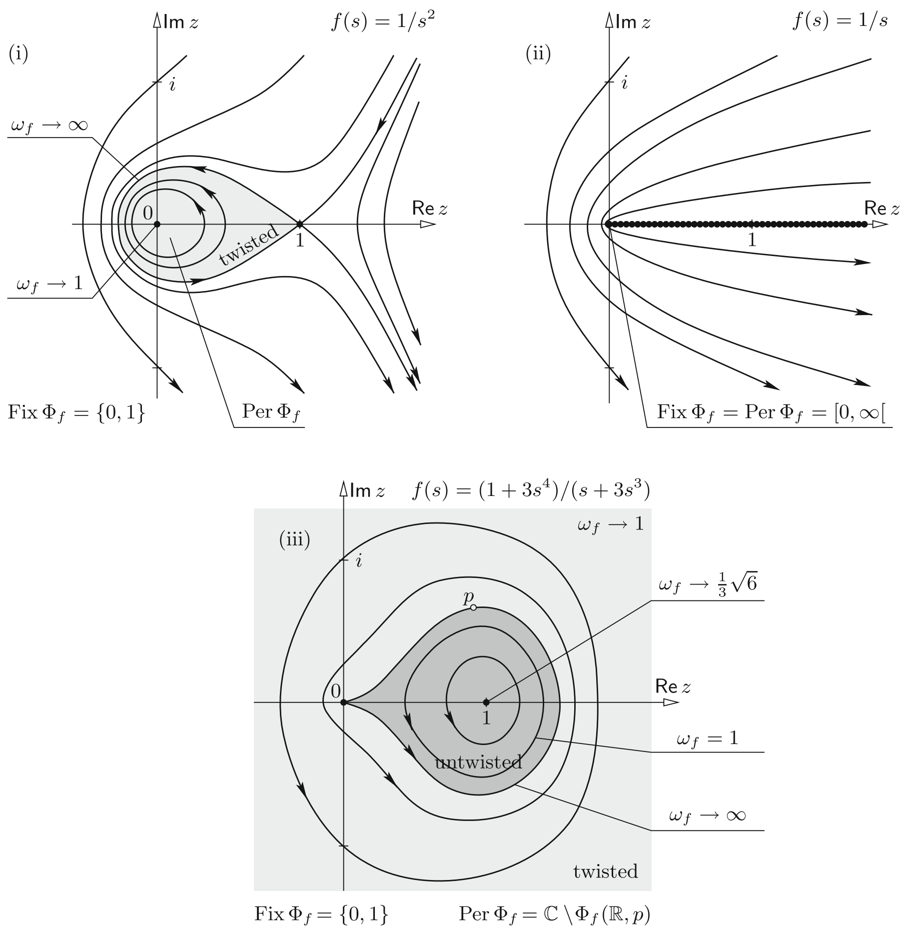

Fig. 3 When 0 is a fixed point of Φ_f, it may be a center (i), non-isolated (ii), or isolated (iii); see Example 2.12

Remark 2.13 (i) Note that $f(s) f'(s) \neq 0$ for all $s \in \mathbb{R}^+$, and hence Lemma 2.9(iii) applies in most examples above, the only exception being Example 2.12(iii).

(ii) If $0 \in$ Fix Φ_f then 0 can be neither a (topological) saddle nor a source or sink, due to (2.2) and the fact that Φ_f preserves Lebesgue measure on $\mathbb{C}$. Rather, the fixed point 0 must be a center or degenerate, as in Example 2.12(i) or (ii,iii), respectively.

3 Characterizing Closed Solutions of $\kappa = f(r)$

In all that follows, let $\mathbb{J}_c \subset \mathbb{R}$ be a non-empty open interval with $0 \in \mathbb{J}_c$ and $c : \mathbb{J}_c \to \mathbb{C}$ a smooth path parametrized by arc length, that is, $|\dot{c}(t)| = 1$ for all $t \in \mathbb{J}_c$. For every $Q \in \mathrm{O}(2)$ let $c_Q(t) = Q \circ c(\epsilon_Q t)$ for all $t \in \mathbb{J}_{c_Q} = \epsilon_Q \mathbb{J}_c$. Thus c_Q is either a rotated (if $\epsilon_Q = 1$) or a reflection-reversed (if $\epsilon_Q = -1$) copy of c. Two smooth paths $c, \widehat{c}$ parametrized by arc length are **equivalent** if $\mathbb{J}_{\widehat{c}} = a + \mathbb{J}_c$ for some $a \in \mathbb{R}$ and $\widehat{c}(t) = c(t - a)$ for all $t \in \mathbb{J}_{\widehat{c}}$. Refer to any equivalence class as an **oriented smooth curve** $\mathcal{C}$, and let $[\mathcal{C}] = c(\mathbb{J}_c)$ where $c \in \mathcal{C}$; thus $[\mathcal{C}] \subset \mathbb{C}$ simply is the set of points parametrized by some (and hence any) $c \in \mathcal{C}$. Also, for every $Q \in \mathrm{O}(2)$ let $\mathcal{C}_Q$ be the equivalence class of c_Q for some $c \in \mathcal{C}$. Note that $\widehat{\mathcal{C}} = \mathcal{C}_Q$ implies $\left[\widehat{\mathcal{C}}\right] = Q([\mathcal{C}])$, but the converse is not true in general.

Given any $c : \mathbb{J}_c \to \mathbb{C}$, associate with it a smooth function $\vartheta_c : \mathbb{J}_c \to \mathbb{R}$ such that $\dot{c} = e^{i\vartheta_c}$. Clearly, ϑ_c is determined by c only up to an additive integer multiple of 2π. Recall that the curvature of c is $\kappa_c = \dot{\vartheta}_c$. If $c, \widehat{c}$ are equivalent then $\kappa_{\widehat{c}}(t) = \kappa_c(t-a)$ for all $t \in \mathbb{J}_{\widehat{c}}$. It makes sense, therefore, so say that, given any smooth function $f : \mathbb{R}^+ \to \mathbb{R}$, the oriented smooth curve $\mathcal{C}$ is a **solution** of

$$\kappa = f(r) \tag{3.1}$$

if $\kappa_c(t) = f(|c(t)|)$ for all $t \in \mathbb{J}_c$ with $c(t) \neq 0$, where c is some (and hence any) element of $\mathcal{C}$. A solution $\mathcal{C}$ of (3.1) is **maximal** if the set $[\mathcal{C}]$ cannot be enlarged any further, that is, if $[\mathcal{C}] \subset \left[\widehat{\mathcal{C}}\right]$ for any solution $\widehat{\mathcal{C}}$ of (3.1) necessarily implies that $[\mathcal{C}] = \left[\widehat{\mathcal{C}}\right]$; see, e.g., [34, Sec. 1]. Note that $\mathcal{C}$ is a (maximal) solution of (3.1) if and only if $\mathcal{C}_Q$ is a (maximal) solution for every $Q \in \mathrm{O}(2)$. Also, given any $p \in \mathbb{C}_\times$ and $\vartheta \in \mathbb{R}$, there exists a (locally unique) maximal solution $\mathcal{C}$ of (3.1) such that $c(0) = p$ and $\vartheta_c(0) = \vartheta$ for some $c \in \mathcal{C}$.

As alluded to already in the Introduction, the main objective of this article is to systematically study all (maximal) solutions of (3.1). This is accomplished by making these solutions correspond to the orbits of the planar flow Φ_f introduced in the previous section. To establish such a correspondance, notice first that for a smooth path $c : \mathbb{J}_c \to \mathbb{C}$ parametrized by arc length, (3.1) with $p = e^{i\vartheta} \in S^1$ simply reads

$$\dot{c} = p, \quad \dot{p} = if(|c|)p,$$

provided that $c \neq 0$. While this may be read as an ODE on $\mathbb{C}_\times \times S^1$, the dimension of the latter phase space can actually be reduced with very little effort. Specifically, given c, associate with it the smooth path $z_c : \mathbb{J}_c \to \mathbb{C}$ with

$$z_c(t) = \overline{ic(t)e^{-i\vartheta_c(t)}} = -i\,\overline{c(t)}e^{i\vartheta_c(t)} \quad \forall t \in \mathbb{J}_c. \tag{3.2}$$

Note that $|z_c| = |c|$. Now, assume that the oriented smooth curve $\mathcal{C}$ is a solution of (3.1), and pick any $c \in \mathcal{C}$. Differentiation of (3.2) yields, for $t \in \mathbb{J}_c$,

$$\dot{z}_c(t) = -i\,\overline{\dot{c}(t)}e^{i\vartheta_c(t)} - i\,\overline{c(t)}e^{i\vartheta_c(t)}i\kappa_c(t) = -i + iz_c(t)f(|z_c(t)|),$$

provided that $c(t) \neq 0$. At least on the non-empty open set $\{t \in \mathbb{J}_c : c(t) \neq 0\}$, therefore, z_c is a solution of (2.1), an ODE on $\mathbb{C}_\times$. It is the purpose of this section to demonstrate that the correspondance $c \leftrightarrow z_c$ indeed enables the systematic study of (3.1) by way of Φ_f. A first simple observation in this regard is that most solutions of (3.1) can be reconstructed from the corresponding orbit of Φ_f; the routine proof is left to the interested reader, as are the proofs of several equally elementary observations below.

Proposition 3.1 *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be smooth. For every $p \in \mathbb{C} \setminus \Phi_f(\mathbb{R}, 0)$ and $\vartheta \in \mathbb{R}$, the smooth path $c : \mathbb{R} \to \mathbb{C}_\times$ given by*

$$c(t) = |p|e^{i\vartheta + i\int_0^t \mathrm{d}u/\overline{\Phi_f(u,p)}} \quad \forall t \in \mathbb{R},$$

is parametrized by arc length, satisfies $\kappa_c = f(|c|)$, and $z_c(t) = \Phi_f(t, p)$ for all $t \in \mathbb{R}$.

Remark 3.2 Arguably, (3.2) might be more natural still if it were made the definition of $\overline{z_c}$ rather than of z_c, a modification that would not affect the substance of the subsequent analysis. However, the specific form of (3.2) has been chosen in order to ensure that, in all relevant situations, the paths c and z_c have the same orientation; see, for instance, the proof of Lemma 3.10 below.

Another basic observation is that replacing c with $\widehat{c}_Q$, where $\widehat{c}$ is equivalent to c and $Q \in \mathrm{O}(2)$, affects z_c only in a trivial way.

Proposition 3.3 *Let $\mathcal{C}$ be an oriented smooth curve, and $c, \widehat{c} \in \mathcal{C}$. For every $Q \in \mathrm{O}(2)$ there exists $a \in \mathbb{R}$ such that*

$$z_{\widehat{c}_Q}(t) = \begin{cases} z_c(t-a) \text{ if } \epsilon_Q = 1, \\ \overline{z_c(a-t)} \text{ if } \epsilon_Q = -1, \end{cases}$$

for all $t \in \mathbb{J}_{\widehat{c}_Q} = a + \epsilon_Q \mathbb{J}$.

With Propositions 2.1 and 3.3, every solution $\mathcal{C}$ of (3.1) corresponds to a uniquely determined Φ_f-orbit, namely $\Phi_f\big(\mathbb{R}, z_c(0)\big)$, and it is easily seen that every Φ_f-orbit, with the possible exception of $\Phi_f(\mathbb{R}, 0)$, can be obtained that way. Moreover, by Proposition 3.3 a Φ_f-conjugate orbit is obtained if $\mathcal{C}$ is replaced by $\mathcal{C}_Q$ for any $Q \in \mathrm{O}(2)$. Leaving aside trivial exceptions, a stronger statement can be made that has the additional benefit of being reversible.

Proposition 3.4 *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be smooth with $\sup\{f \neq 0\} = \infty$. For every two maximal solutions $\mathcal{C}, \widehat{\mathcal{C}}$ of (3.1) the following are equivalent:*

(i) $[\mathcal{C}], \big[\widehat{\mathcal{C}}\big]$ *are* $\mathrm{O}(2)$*-congruent;*
(ii) $z_c(0), z_{\widehat{c}}(0)$ *are Φ_f-conjugate for some (and hence every) $c \in \mathcal{C}$, $\widehat{c} \in \widehat{\mathcal{C}}$.*

Whenever $\{f \neq 0\}$ is unbounded, therefore, Proposition 3.4 establishes a bijection between the maximal solutions of (3.1) modulo rotations and reflection-reversals on the one hand, and the orbits of Φ_f modulo Φ_f-conjugacy on the other hand. As a consequence, the study of maximal solutions of (3.1) modulo $\mathrm{O}(2)$-congruence, the central theme throughout the remainder of the present article, can proceed mostly via a careful analysis of the orbits of Φ_f.

Remark 3.5 Maximality of $\mathcal{C}, \widehat{\mathcal{C}}$ is essential for both implications in Proposition 3.4. Moreover, (i)⇒(ii) may fail when $\{f \neq 0\}$ is bounded, whereas (ii)⇒(i) remains correct in this case also.

Say that an oriented smooth curve $\mathcal{C}$ is **closed** if $[\mathcal{C}]$ is compact, and **simple closed** if $[\mathcal{C}]$ is homeomorphic to the unit circle. Though not without precursors in the literature, e.g., in [28], this terminology is tailor-made for the present article and may seem unconventional. For maximal solutions of (3.1), though, it is easily seen to be equivalent to a more conventional notion [10, 17, 27].

Proposition 3.6 *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be smooth. For every maximal solution $\mathcal{C}$ of (3.1) the following are equivalent:*

(i) there exists an oriented smooth curve $\widehat{\mathcal{C}}$ and $a \in \mathbb{R}^+$ with $\big[\widehat{\mathcal{C}}\big] = [\mathcal{C}]$ and $\widehat{c}(t+a) = \widehat{c}(t)$ for some $\widehat{c} \in \widehat{\mathcal{C}}$ and all $t \in \mathbb{J}_{\widehat{c}} = \mathbb{R}$;
(ii) $\mathcal{C}$ is closed.

Moreover, $\mathcal{C}$ is simple closed if and only if $\widehat{c}$ in (i) can be chosen to be one-to-one on $[0, a[$.

Remark 3.7 Plainly, (i)⇒(ii) in Proposition 3.6 for every oriented smooth curve $\mathcal{C}$. However, (ii)⇒(i) may fail when $\mathcal{C}$ is not a maximal solution of (3.1).

Notice that if two closed solutions $\mathcal{C}, \widehat{\mathcal{C}}$ of (3.1) are O(2)-congruent then in fact $[\mathcal{C}] = e^{i\vartheta}[\widehat{\mathcal{C}}]$ for some $\vartheta \in \mathbb{R}$. The main goal of the remainder of this section is to characterize maximal solutions that are closed, perhaps even simple closed curves. As the reader may have suspected all along, the planar flow Φ_f and in particular the net winding ω_f of its periodic points are instrumental in this characterization.

Lemma 3.8 *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be smooth. For every maximal solution $\mathcal{C}$ of* (3.1) *with* $\operatorname{dist}(0, [\mathcal{C}]) > 0$ *the following are equivalent:*

(i) $\mathcal{C}$ is closed;
(ii) $z_c(0) \in \operatorname{Per}\Phi_f$ and $\omega_f\big(z_c(0)\big) \in \mathbb{Q}$ for some (and hence every) $c \in \mathcal{C}$.

Proof To prove (i)⇒(ii), let $\mathcal{C}$ be a closed maximal solution of (3.1), and $c \in \mathcal{C}$. By Propositions 3.4 and 3.6, it can be assumed that $c(t+a) = c(t)$ for some $a \in \mathbb{R}^+$ and all $t \in \mathbb{J}_c = \mathbb{R}$. Differentiation yields $\vartheta_c(t+a) - \vartheta_c(t) = 2\pi k$ for some $k \in \mathbb{Z}$ and all $t \in \mathbb{R}$. From (3.2), it is clear that $z_c(t+a) = z_c(t)$, and hence $z_c(0) \in \operatorname{Per}\Phi_f$. Since obviously $\omega_f\big(z_c(0)\big) = 0 \in \mathbb{Q}$ whenever $z_c(0) \in \operatorname{Fix}\Phi_f$, henceforth assume that $z_c(0) \in \operatorname{Per}\Phi_f \setminus \operatorname{Fix}\Phi_f$, in which case $a = mT_f\big(z_c(0)\big)$ for some $m \in \mathbb{N}$. Moreover, since $\kappa_c(t) = f(|c(t)|)$ for almost all t,

$$\begin{aligned} 2\pi k &= \int_0^a \dot{\vartheta}_c(t)\,\mathrm{d}t = \int_0^a f(|z_c(t)|)\,\mathrm{d}t \\ &= m\int_0^{T_f(z_c(0))} f\big(\big|\Phi_f\big(t, z_c(0)\big)\big|\big)\,\mathrm{d}t = \pm 2\pi m\omega_f\big(z_c(0)\big), \end{aligned} \tag{3.3}$$

and so $\omega_f\big(z_c(0)\big) \in \mathbb{Q}$, as claimed.

To prove (ii)⇒(i), assume that $z_c(0) \in \operatorname{Per}\Phi_f$ and $\omega_f\big(z_c(0)\big) \in \mathbb{Q}$. Note that $0 \notin \Phi_f\big(\mathbb{R}, z_c(0)\big)$ since otherwise $\operatorname{dist}(0, [\mathcal{C}]) = 0$. If $z_c(0) \in \operatorname{Fix}\Phi_f$ then $[\mathcal{C}]$ equals the circle with radius $|z_c(0)| > 0$ centered at 0, so clearly $\mathcal{C}$ is closed. Assume from now on that $z_c(0) \in \operatorname{Per}\Phi_f \setminus \operatorname{Fix}\Phi_f$, with $b := \frac{1}{2}T_f\big(z_c(0)\big) > 0$ for convenience. Also, let $p = z_c(0)$, and pick $\vartheta \in \mathbb{R}$ such that $c(0) = |p|e^{i\vartheta}$. (This is possible because $|c(0)| = |p|$.) By Proposition 3.1, the smooth path $\widehat{c} : \mathbb{R} \to \mathbb{C}$ given by

$$\widehat{c}(t) = |p|e^{i\vartheta + i\int_0^t \mathrm{d}u/\overline{\Phi_f(u,p)}} \quad \forall t \in \mathbb{R},$$

is parametrized by arc length, and $\kappa_{\widehat{c}} = f(|\widehat{c}|)$. Moreover, $\widehat{c}(0) = c(0)$, $\dot{\widehat{c}}(0) = \dot{c}(0)$, and consequently $c(\mathbb{J}_c) \subset \widehat{c}(\mathbb{R})$. By maximality, $[\mathcal{C}] = \widehat{c}(\mathbb{R})$, so it suffices to show that $\widehat{c}(\mathbb{R})$ is compact. To this end, simply notice that by (2.6),

$$\widehat{c}(t+2b) = \widehat{c}(t)e^{i\int_0^{2b}\mathrm{d}u/\overline{\Phi_f(u,p)}} = \widehat{c}(t)e^{\pm 2\pi i(\omega_f(p) - k_p)} = \widehat{c}(t)e^{\pm 2\pi i\omega_f(p)} \quad \forall t \in \mathbb{R}.$$

Picking $n \in \mathbb{N}$ such that $n\omega_f(p) \in \mathbb{Z}$ yields $\widehat{c}(t+2nb) = \widehat{c}(t)$ for all $t \in \mathbb{R}$. Thus $\widehat{c}(\mathbb{R}) = \widehat{c}([0, 2nb])$ indeed is compact, and $\mathcal{C}$ is closed. □

Remark 3.9 As can be seen from the above proof, the assumption $\operatorname{dist}(0, [\mathcal{C}]) > 0$ is not needed for (i)⇒(ii). By contrast, (ii)⇒(i) may fail without it. Notice, however, that (i)⇔(ii) for *every* maximal solution $\mathcal{C}$ of (3.1) provided that f can be extended smoothly to $s = 0$, e.g., if f is a polynomial. In this case, $\mathcal{C} \leftrightarrow \Phi_f\big(\mathbb{R}, z_c(0)\big)$ establishes a bijection between the closed maximal solutions of (3.1) modulo rotations and those periodic orbits of Φ_f whose net winding is a rational number.

Let $\mathcal{C}$ be a closed maximal solution of (3.1). By Proposition 3.3 and Lemma 3.8, $z_c(0)$ is contained in the same Φ_f-orbit for every $c \in \mathcal{C}$. It makes sense, therefore, to refer to $\mathcal{C}$ as being (un)twisted whenever $\Phi_f\big(\mathbb{R}, z_c(0)\big)$ is (un)twisted, and to let $\omega_f(\mathcal{C}) = \omega_f\big(z_c(0)\big)$ for any $c \in \mathcal{C}$. Not too surprisingly, if $\mathcal{C}$ is simple closed then the possible values of $\omega_f(\mathcal{C})$ are severely constrained. Henceforth the term **Jordan solution** is used to refer to any maximal solution of (3.1) that is simple closed. The following observation and its partial converse (Theorem 3.13 below) are the main results of this section.

Lemma 3.10 *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be smooth, and $\mathcal{C}$ a Jordan solution of (3.1).*

(i) If $\mathcal{C}$ is untwisted then $\omega_f(\mathcal{C}) = 0$ or $\omega_f(\mathcal{C}) = 1/n$ for some $n \in \mathbb{N}$.
(ii) If $\mathcal{C}$ is twisted then $\omega_f(\mathcal{C}) = 1$.

Proof Let $\mathcal{C}$ be a Jordan solution, and $c \in \mathcal{C}$. By Proposition 3.6, it may be assumed that $c(t + a) = c(t)$ for some $a \in \mathbb{R}^+$ and all $t \in \mathbb{J}_c = \mathbb{R}$, with c being one-to-one on $[0, a[$. If $z_c(0) \in \operatorname{Fix}\Phi_f$ then $\{z_c(0)\} \neq \{0\}$ clearly is untwisted, and $\omega_f(\mathcal{C}) = 0$, so henceforth assume that $z_c(0) \in \operatorname{Per}\Phi_f \setminus \operatorname{Fix}\Phi_f$.

It will first be shown that $\Phi_f\big(\cdot, z_c(0)\big)$ and c have the same orientation: Either both are oriented counter-clockwise, or both are oriented clockwise. To see this, let $s_1 < s_2$ be the intersection points of $\Phi_f\big(\mathbb{R}, z_c(0)\big)$ with the real axis. Then $H_f(s_1) = H_f(s_2)$, and hence $|s_1| \neq |s_2|$. Assume for instance that $|s_1| < |s_2|$. In this case, $s_2 = \max_{t\in\mathbb{R}} \big|\Phi_f\big(t, z_c(0)\big)\big| = \max_{t\in\mathbb{R}} |c(t)| > 0$. Since $\mathrm{d}|z|/\mathrm{d}t = -\mathsf{Im}\, z/|z| < 0$ whenever $\mathsf{Im}\, z > 0$, and $\dot{z}|_{z=s_2} = i(s_2 f(s_2) - 1)$, necessarily $s_2 f(s_2) > 1$. Thus $f(s_2) > 1/s_2 > 0$, and $\Phi_f\big(\cdot, z_c(0)\big)$ is oriented counter-clockwise. Pick t_0 with $|c(t_0)| = s_2$. Since $|c|$ attains a maximum for $t = t_0$, and $\kappa_c(t_0) = f(s_2) > 0$, clearly c is oriented counter-clockwise also. A completely analogous argument shows that $\Phi_f\big(\cdot, z_c(0)\big)$ and c both are oriented clockwise when $|s_1| > |s_2|$.

With $k \in \mathbb{Z}$ and $m \in \mathbb{N}$ as in the proof of Lemma 3.8, the theorem of turning tangents (see, e.g., [17, Thm. 5.7.2]) yields $k = \pm 1$, and (3.3) simply reads $\pm 2\pi = \pm 2\pi m \omega_f(\mathcal{C})$, where the plus (respectively, minus) signs apply when $\Phi_f\big(\cdot, z_c(0)\big)$ and c are oriented counter-clockwise (respectively, clockwise). Thus $\omega_f(\mathcal{C}) = 1/m$ regardless of whether $\mathcal{C}$ is untwisted or twisted. Clearly, this proves (i).

To prove (ii), let $\mathcal{C}$ be twisted, and assume first that $0 \in [\mathcal{C}]$. Then $\omega_f(\mathcal{C}) = \omega_f(0)$ and $0 \in \operatorname{Per}\Phi_f \setminus \operatorname{Fix}\Phi_f$. Assume w.l.o.g. that $c(0) = 0$, so $z_c(0) = 0$ as well. But $c\big(T_f(0)\big) = 0$ also, and hence $T_f(0) \geq a = mT_f(0) > 0$, yielding $m \leq 1$, that is, $\omega_f(\mathcal{C}) = 1$.

To complete the proof of (ii), let $\mathcal{C}$ be twisted but assume that $0 \notin [\mathcal{C}]$. With the numbers $s_1 < s_2$ as above, this means that $s_1 < 0 < s_2$. Assume for instance that $|s_1| < s_2$, and w.l.o.g. that $c(0) = s_2$ and $\vartheta_c(0) = \frac{1}{2}\pi$. Then $z_c(0) = s_2$, and $\Phi_f(\cdot, s_2)$ is oriented counter-clockwise. As in the proof of Lemma 2.6, write $\Phi_f(t, s_2) = |\Phi_f(t, s_2)|e^{i\varphi(t)}$, where φ is smooth and $\varphi(0) = 0$. As seen there,

$$\dot{\varphi}(t) = f(|z_c(t)|) - \frac{\cos\varphi(t)}{|z_c(t)|} = f(|z_c(t)|) - \frac{\mathsf{Re}\, z_c(t)}{|z_c(t)|^2} \quad \forall t \in \mathbb{R}. \tag{3.4}$$

Deduce from (3.2) that $c(t) = |c(t)|e^{i\alpha(t)}$ for all $t \in \mathbb{R}$, with the smooth function $\alpha = \vartheta_c - \frac{1}{2}\pi - \varphi$. Notice that $\alpha(0) = 0$, and by (3.4),

$$\dot{\alpha}(t) = \dot{\vartheta}_c(t) - \dot{\varphi}(t) = \frac{\mathsf{Re}\, z_c(t)}{|z_c(t)|^2} = \mathsf{Re}\,\frac{1}{z_c(t)} \quad \forall t \in \mathbb{R}. \tag{3.5}$$

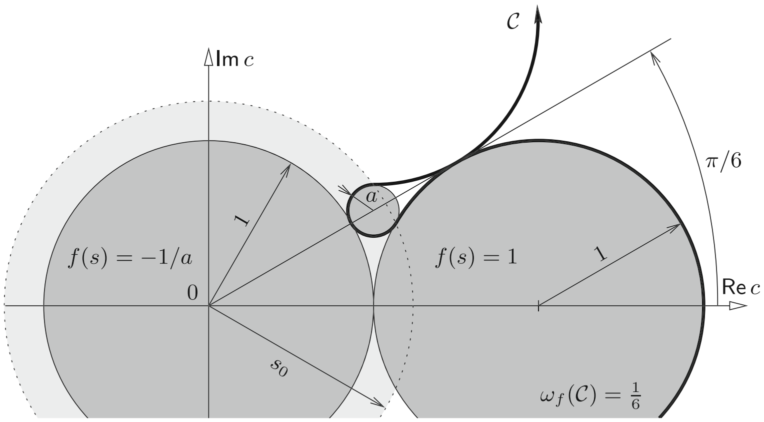

Fig. 4 A closed maximal solution $\mathcal{C}$ of (3.1) with $\omega_f(\mathcal{C}) = \frac{1}{6}$ may or may not be a Jordan solution; see Example 3.11

In particular, $\dot{\alpha}(0) > 0$ and $\dot{\alpha}(b) < 0$, where $b := \frac{1}{2}T_f(s_2) \le \frac{1}{2}a$ for convenience. Also, from the basic symmetry (2.3) it is readily deduced that

$$c(-t) = \overline{c(t)} \qquad \text{and} \qquad c(2b - t) = e^{2i\alpha(b)}\overline{c(t)} \quad \forall t \in \mathbb{R}. \tag{3.6}$$

Now, suppose that $\alpha(t_0) \ge \pi$ or $\alpha(t_0) \le 0$ for some $0 < t_0 < b$. Then $\overline{c(t_0)} = c(t_0)$, and so $c(-t_0) = c(t_0)$, by the left equality in (3.6). Thus c would not be one-to-one on $]-b, b[\subset]-\frac{1}{2}a, \frac{1}{2}a[$, contradicting the initial assumption on c. Consequently, $0 < \alpha(t) < \pi$ for all $0 < t < b$, and $0 \le \alpha(b) \le \pi$. Similarly, if $\alpha(b) > 0$ then $\alpha(t_0) = \alpha(b)$ for some $0 < t_0 < b$, so $e^{2i\alpha(b)}\overline{c(t_0)} = c(t_0)$, and hence $c(2b - t_0) = c(t_0)$, by the right equality in (3.6), leading again to the contradictory conclusion that c is not one-to-one on $]0, 2b[\subset [0, a[$. In summary, therefore, $\alpha(b) = 0$. Utilizing (2.6) and (3.5) yields

$$0 = \alpha(b) = \int_0^{\frac{1}{2}T_f(s_2)} \dot{\alpha}(t)\,\mathrm{d}t = \int_0^{\frac{1}{2}T_f(s_2)} \frac{\mathrm{d}t}{\Phi_f(t, s_2)} = \pi(\omega_f(s_2) - 1),$$

and so $\omega_f(\mathcal{C}) = \omega_f(s_2) = 1$. Since the case $|s_1| > s_2$ is completely analogous, $\omega_f(\mathcal{C}) = 1$ for every twisted Jordan solution $\mathcal{C}$. $\square$

The following two examples illustrate how reversing the conclusion of Lemma 3.10 in general may be delicate: Whether or not an untwisted closed maximal solution $\mathcal{C}$ of (3.1) with $\omega_f(\mathcal{C}) = 1/n$ for some $n \in \mathbb{N}$ actually is a Jordan solution may depend on properties of f not obvious from the outset.

Example 3.11 Consider three touching discs with radii 1, 1, and a, respectively, positioned as shown in Fig. 4 (dark grey), where simple trigonometry yields $a = \frac{2}{3}\sqrt{3} - 1 = 0.1547$ and $s_0 = \sqrt{5 - 2\sqrt{3}} = 1.239$. Let $f(s) = -1/a$ when $0 < s < s_0$ (light grey), and $f(s) = 1$ when $s > s_0$. The closed solution $\mathcal{C}$ indicated in Fig. 4 is not simple, and yet $\omega_f(\mathcal{C}) = \frac{1}{6}$. Of course, f is not continuous, but given any $\varepsilon > 0$, it is straightforward to construct smooth functions $f_\varepsilon, \widehat{f_\varepsilon} : \mathbb{R}^+ \to \mathbb{R}$ with $\|f_\varepsilon - \widehat{f_\varepsilon}\|_\infty + \|f'_\varepsilon - \widehat{f}'_\varepsilon\|_\infty < \varepsilon$, and corresponding closed maximal solutions $\mathcal{C}_\varepsilon, \widehat{\mathcal{C}_\varepsilon}$ with $\omega_{f_\varepsilon}(\mathcal{C}_\varepsilon) = \omega_{\widehat{f_\varepsilon}}(\widehat{\mathcal{C}_\varepsilon}) = \frac{1}{6}$ such that $\mathcal{C}_\varepsilon$ is simple closed whereas $\widehat{\mathcal{C}_\varepsilon}$ is not.

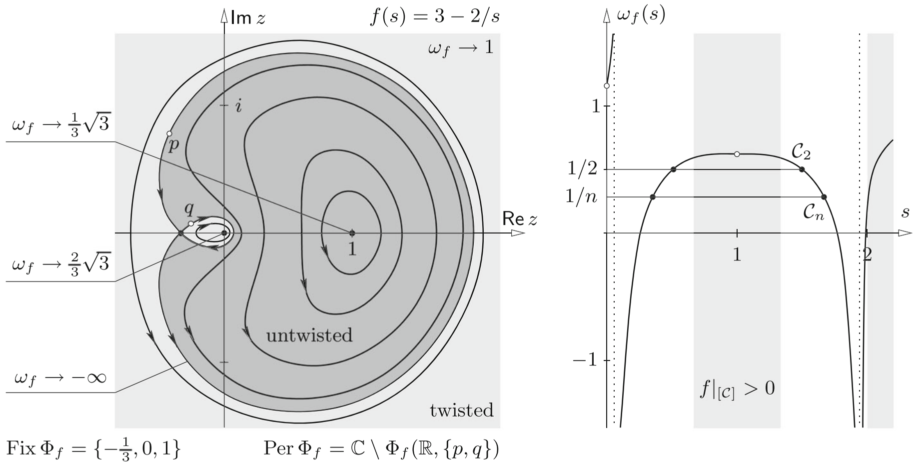

Fig. 5 For $f(s) = 3 - 2/s$ every point not on the two homoclinic loops associated with the saddle $-\frac{1}{3}$ is periodic, with the centers 0 and 1 being the only other fixed points (left); see Example 3.12. Although (3.1) has, for every $n \in \mathbb{N} \setminus \{1\}$, an untwisted closed maximal solution $\mathcal{C}_n$ with $\omega_f(\mathcal{C}_n) = 1/n$, only finitely many $\mathcal{C}_n$ are Jordan solutions, and none are convex (right)

Example 3.12 Let $f(s) = 3-2/s$ for all $s \in \mathbb{R}^+$. Note that $f'(s) = 2/s^2 > 0$ and $F''(s) = 3$ for all $s \in \mathbb{R}^+$. The flow Φ_f has exactly three fixed points: $-\frac{1}{3}$ which is a saddle with two associated homoclinic loops given implicitly by $6\,\mathsf{Re}\,z = 9|z|^2 - 12|z| + 1$, and two centers 0 and 1, with $\lim_{z\to 0}\omega_f(z) = \frac{2}{3}\sqrt{3} > 1$ and $\lim_{z\to 1}\omega_f(z) = \frac{1}{3}\sqrt{3} < 1$. Every point not on the two homoclinic loops is periodic, and $\lim_{z\to\infty}\omega_f(z) = 1$. Note that $\omega_f(z) \neq 1$ for every $z \in$ Per Φ_f, by Lemma 2.9(ii). From the phase portrait of Φ_f in Fig. 5, it is clear that $\omega_f(z) \to -\infty$ (respectively, $\omega_f(z) \to \infty$) as z approaches a homoclinic loop from within the untwisted or outer twisted (respectively, inner twisted) regions. Since $\frac{1}{3}\sqrt{3} > \frac{1}{2}$, for every $n \in \mathbb{N} \setminus \{1\}$ there exists at least one untwisted closed maximal solution $\mathcal{C}_n$ of (3.1) with $\omega_f(\mathcal{C}_n) = 1/n$. Numerical evidence strongly suggests that $\omega_f(s) > \frac{1}{2}$ for all $\frac{2}{3} \leq s \leq \frac{4}{3}$, and hence the sign of f changes along each $\mathcal{C}_n$. In other words, $\mathcal{C}_n$ is not convex for any n, by [28, Thm. 2.31]. Moreover, it is not hard to see that $\mathcal{C}_n$ is not a simple closed curve if n is large. In this example, therefore, (3.1) has, modulo rotations, only a finite number of Jordan solutions, of which only the two circles with radii $\frac{1}{3}$ and 1, centered at 0 and oriented clockwise and counter-clockwise, respectively, are convex. Rigorously determining the precise number of non-circular Jordan solutions may be a delicate task.

The above examples make it clear that for the conclusion of Lemma 3.10 to be reversed in any generality, additional, possibly rather restrictive assumptions on f have to be imposed. While many assumptions are conceivable in this regard, one clearly suggesting itself through Lemma 2.9 is that

$$f(s)f'(s) \neq 0 \quad \forall s \in \mathbb{R}^+. \tag{3.7}$$

For example, the monomial $f_b(s) = s^b$, with $b \in \mathbb{R}$, to be studied in detail in Sect. 5 below, satisfies (3.7) for every $b \neq 0$. Assuming (3.7), the conclusion of Lemma 3.10 can be strengthened and reversed rather neatly.

Theorem 3.13 *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be smooth and satisfy (3.7). For every closed maximal solution $\mathcal{C}$ of (3.1) the following are equivalent:*

(i) $\mathcal{C}$ *is untwisted, and* $\omega_f(\mathcal{C}) = 0$ *or* $\omega_f(\mathcal{C}) = 1/n$ *for some* $n \in \mathbb{N} \setminus \{1\}$*;*
(ii) $\mathcal{C}$ *is a Jordan solution.*

Moreover, $\omega_f(\mathcal{C}) = 0$ *if and only if* $\mathcal{C}$ *is a circle of radius* $|z|$ *centered at* 0*, with* $z \in \mathbb{F}_f$*, oriented counter-clockwise when* $z > 0$*, and clockwise when* $z < 0$*.*

Proof Throughout, let $\mathcal{C}$ be a closed maximal solution, and $c \in \mathcal{C}$ and $c(t+a) = c(t)$ for some $a \in \mathbb{R}^+$ and all $t \in \mathbb{J}_c = \mathbb{R}$. By Remark 2.3(iii) assume w.l.o.g. that $f > 0$.

To prove (i)⇒(ii), recall from Lemma 3.8 that $z_c(0) \in \operatorname{Per}\Phi_f$. (This part of the lemma does not require $0 \notin [\mathcal{C}]$; see Remark 3.9.) By Lemma 2.9, $\omega_f(\mathcal{C}) > 0$ unless $z_c(0) \in \operatorname{Fix}\Phi_f$, in which case $z_c(0) \neq 0$, and $\mathcal{C}$ is a circle with radius $|z_c(0)|$ centered at 0, hence obviously a Jordan solution; since $\kappa_c = f(|z_c(0)|) = 1/z_c(0)$, this circle is oriented counter-clockwise when $z_c(0) > 0$, and clockwise when $z_c(0) < 0$. Thus it only remains to consider the case $\omega_f(\mathcal{C}) = 1/n$ where necessarily $z_c(0) \in \operatorname{Per}\Phi_f \setminus \operatorname{Fix}\Phi_f$. As in the proof of Lemma 3.10, let $s_1 < s_2$ be the intersection points of $\Phi_f\big(\mathbb{R}, z_c(0)\big)$ with the real axis, and assume w.l.o.g. that $z_c(0) = s_2$. Noticing that $s_1 s_2 > 0$ since $\mathcal{C}$ is untwisted, assume, for instance, that $0 < s_1 < s_2$. Since $\operatorname{Re}\dot{z}_c = -\operatorname{Im} z_c f(|z_c|)$ is negative (respectively, positive) whenever $\operatorname{Im} z_c > 0$ (respectively, $\operatorname{Im} z_c < 0$), clearly $\operatorname{Re}\Phi_f(t, s_2) \in [s_1, s_2]$ for all $t \in \mathbb{R}$. Also, writing $c(t) = |c(t)|e^{i\alpha(t)}$ with the smooth function α satisfying $\alpha(0) = 0$ yields

$$\dot{\alpha}(t) = \frac{\operatorname{Im}\big(\dot{c}(t)\overline{c(t)}\big)}{|c(t)|^2} = \frac{\operatorname{Re}\Phi_f(t, s_2)}{|\Phi_f(t, s_2)|^2} = \operatorname{Re}\frac{1}{\Phi_f(t, s_2)} > 0 \quad \forall t \in \mathbb{R},$$

in accordance with (3.5). Thus α is increasing, and with $b := \frac{1}{2}T_f(s_2)$ for convenience,

$$\begin{aligned}\alpha(t+2b) - \alpha(t) &= \operatorname{Re}\int_t^{t+2b} \frac{\mathrm{d}u}{\Phi_f(u, s_2)} \\ &= \int_0^{T_f(s_2)} \frac{\mathrm{d}u}{\Phi_f(u, s_2)} = 2\pi\omega_f(s_2) = \frac{2\pi}{n} \quad \forall t \in \mathbb{R},\end{aligned}$$

as well as $\alpha(b) = \pi/n$. It follows that c is one-to-one on $[0, 2nb[$, but also

$$c(t+2nb) = |z_c(t+2nb)|e^{i\alpha(t+2nb)} = |z_c(t)|e^{i\alpha(t)} = c(t) \quad \forall t \in \mathbb{R}.$$

Thus $\mathcal{C}$ is a Jordan solution. An analogous argument for the case $s_1 < s_2 < 0$ completes the proof of (i)⇒(ii).

To see that (ii)⇒(i) simply recall from Lemma 2.9 that (3.7) implies $\omega_f(\mathcal{C}) \neq 1$, and hence the claim immediately follows from Lemma 3.10. □

Remark 3.14 The reader may have noticed that the proof of Lemma 3.10 presented above does contain a possible, if rather unpractical *necessary and sufficient condition* for a closed maximal solution to be Jordan. To illustrate the simple idea, let $\mathcal{C}$ be, for instance, a (non-circular) untwisted closed maximal solution, with associated numbers $0 < s_1 < s_2$ as in the proof of Lemma 3.10. Define $\omega_f^* : \big]0, \frac{1}{2}T_f(s_2)\big[\to \mathbb{R}$ by

$$\omega_f^*(t) = \pm\frac{1}{\pi}\operatorname{Re}\int_0^t \frac{\mathrm{d}u}{\Phi_f(u, s_2)} \quad \forall 0 < t < \frac{1}{2}T_f(s_2),$$

with the sign as in (2.5) and (2.6). Notice that $\omega_f^*(0+) = 0$ and $\omega_f^*\big(\frac{1}{2}T_f(s_2)-\big) = \omega_f(\mathcal{C})$; in fact, by (3.5) simply $\omega_f^*(t) = \pm\alpha(t)/\pi$. With this, it is clear that $\mathcal{C}$ is a Jordan solution if and only if $\omega_f(\mathcal{C}) = 1/n$ for some $n \in \mathbb{N}$, and

$$0 < \omega_f^*(t) < \omega_f(\mathcal{C}) \quad \forall 0 < t < \tfrac{1}{2}T_f(s_2). \tag{3.8}$$

Notice that in the setting of Theorem 3.13, i.e., with f satisfying (3.7), the function ω_f^* is increasing, as seen in the proof, and hence the otherwise unwieldy condition (3.8) holds automatically.

4 Jordan Solutions and Monotone Net Winding

By Lemma 3.8, closed (maximal) solutions of (3.1) are plentiful unless ω_f is constant on $\operatorname{Per}\Phi_f \setminus \operatorname{Fix}\Phi_f$. By contrast, Lemma 3.10 and Example 3.12 suggest that Jordan solutions are much rarer. In fact, as detailed in this section and the next, Jordan solutions of (3.1) are exceedingly rare for many f. Though much of the analysis could be carried out, at least locally, for far more general f, assume from now on that the smooth function $f : \mathbb{R}^+ \to \mathbb{R}$ satisfies (3.7). This allows Lemma 2.9 and Theorem 3.13 to be applied together. By Theorem 3.13, clearly (3.1) has at least as many different (circular) Jordan solutions as $\mathbb{F}_f$ has elements. For instance, $\mathbb{F}_f = \mathbb{R}^+$ for $f(s) = 1/s$, and correspondingly every (counter-clockwise oriented) circle centered at 0 is a Jordan solution; see Example 2.12(ii). To rule out degenerate situations like this, and thus to make the ultimate results particularly complete and transparent, assume in addition that $F''(s) \neq 0$ for all $s \in \mathbb{R}^+$, so $\mathbb{F}_f$ is either empty or a singleton. For convenience, then, let

$$\mathcal{F} = \left\{ f : \mathbb{R}^+ \to \mathbb{R} \quad \text{is smooth with } f(s)f'(s)F''(s) \neq 0 \,\forall s \in \mathbb{R}^+ \right\}.$$

Plainly, $af \in \mathcal{F}$ for every $a \in \mathbb{R} \setminus \{0\}$ and $f \in \mathcal{F}$. Given $f \in \mathcal{F}$ let $\epsilon_f = \pm 1$ be such that $\epsilon_f F'' > 0$, and notice that the open interval

$$\mathbb{I}_f = \left]\lim\nolimits_{s\to 0} \epsilon_f s f(s), \lim\nolimits_{s\to\infty} \epsilon_f s f(s)\right[$$

is well-defined, non-empty, and does not contain 0. For example, with f from Examples 2.10, 2.11, and 2.12(ii), clearly $f \in \mathcal{F}$, and $\mathbb{I}_f$ equals $]0, 1[$, $\mathbb{R}^+$, and $-\mathbb{R}^+$, respectively. Every open interval contained in $\mathbb{R} \setminus \{0\}$ equals $\mathbb{I}_f$ for an appropriate $f \in \mathcal{F}$. To analyze (3.1) with $f \in \mathcal{F}$, it is convenient to distinguish four cases, depending on the position of $\mathbb{I}_f$ relative to the two-point set $\{-1, 1\}$. Three of the four cases are straightforward, as recorded in Propositions 4.1 to 4.3 below.

Proposition 4.1 *Let $f \in \mathcal{F}$, and assume that $\mathbb{I}_f \cap [-1, 1] = \varnothing$. Then every closed maximal solution of* (3.1) *is twisted; in particular,* (3.1) *has no Jordan solution.*

It is easy to see that with f as in Proposition 4.1, every periodic orbit of Φ_f is twisted, and the center 0 is the only fixed point. Moreover, $\operatorname{Per}\Phi_f = \mathbb{C}$, except perhaps when $\sup \mathbb{I}_f = -1$, in which case Φ_f may have non-periodic points; see Fig. 6.

Proposition 4.2 *Let $f \in \mathcal{F}$, and assume that $\mathbb{I}_f \subset]-1, 1[$. Then every maximal solution of* (3.1) *is unbounded; in particular,* (3.1) *has no Jordan solution.*

Again, with f as in Proposition 4.2, it is easy to see that all orbits of Φ_f are unbounded, and $\operatorname{Per}\Phi_f = \varnothing$, except perhaps when $\inf \mathbb{I}_f = -1$, in which case 0 may be a fixed point; see Fig. 7 and also Example 2.10.

Proposition 4.3 *Let $f \in \mathcal{F}$, and assume that $-1 \in \mathbb{I}_f$. Then $\mathbb{F}_f = \{s\}$, with $s \in \mathbb{R} \setminus \{0\}$ uniquely determined by $sf(|s|) = 1$. Every closed maximal solution of* (3.1) *either is twisted, or else equals the circle with radius $|s|$ centered at* 0*, oriented counter-clockwise when $s > 0$, and clockwise when $s < 0$; in particular,* (3.1) *has exactly one Jordan solution (namely, that circle).*

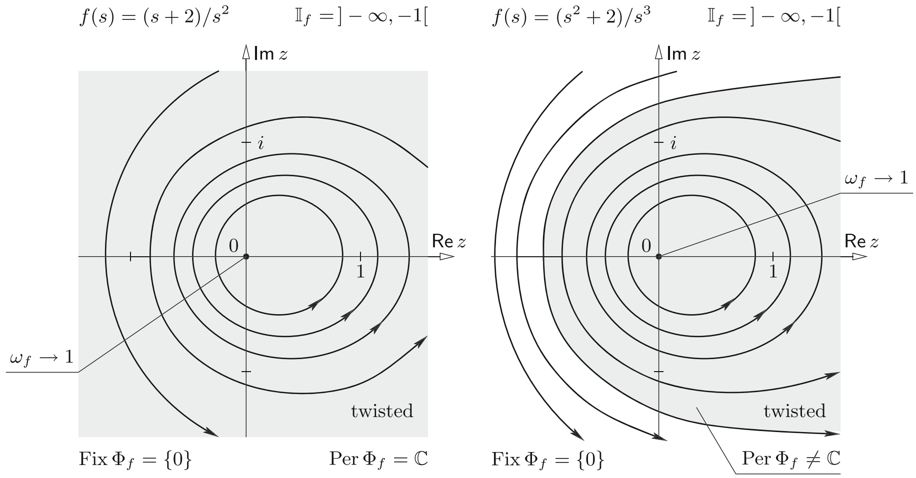

Fig. 6 If $f \in \mathcal{F}$ and $\mathbb{I}_f \cap [-1, 1] = \varnothing$ then every periodic orbit of Φ_f is twisted (left); only when $\sup \mathbb{I}_f = -1$ may non-periodic orbits exist (right)

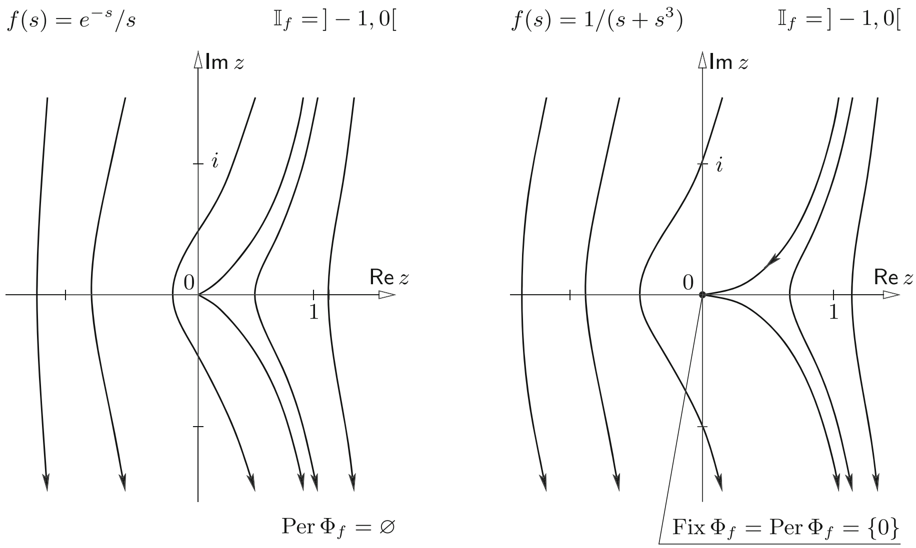

Fig. 7 If $f \in \mathcal{F}$ and $\mathbb{I}_f \subset]-1, 1[$ then Per $\Phi_f = \varnothing$ (left), except when $\inf \mathbb{I}_f = -1$ where 0 may be a fixed point (right)

Note that under the assumptions of Proposition 4.3 the fixed point s is a saddle, its associated homoclinic loop containing the center 0 and all periodic orbits, each of which is twisted; see Example 2.12(i) for a typical phase portrait.

By Propositions 4.1 to 4.3, non-circular Jordan solutions of (3.1) with $f \in \mathcal{F}$, if at all existant, can be found only in the remaining (fourth) case, that is, when $1 \in \mathbb{I}_f$. Again, for convenience let

$$\mathcal{F}^* = \{f \in \mathcal{F} : 1 \in \mathbb{I}_f\}.$$

For example, $f_b \in \mathcal{F}$ for every $b \in \mathbb{R} \setminus \{-1, 0\}$, but $f_b \in \mathcal{F}^*$ only when $b > -1$. For every $f \in \mathcal{F}^*$, note that $\epsilon_f f > 0$, i.e., ϵ_f simply is the (constant) sign of f. Also, $\mathbb{F}_f = \{\epsilon_f s_f\} = \operatorname{Fix} \Phi_f$, where $s_f \in \mathbb{R}^+$ is the unique solution of $sf(s) = \epsilon_f$. The sole fixed point of Φ_f is a center, and $\operatorname{Per} \Phi_f = \mathbb{C}$; see Example 2.11 for a typical phase portrait. Observe that every (periodic) orbit not intersecting the line segment $\epsilon_f\ ]0, s_f]$ is twisted. With Proposition 3.4 and Theorem 3.13, therefore, modulo rotations all maximal solutions of (3.1) with $f \in \mathcal{F}^*$ that could potentially be Jordan solutions are parametrized by that segment. More formally, given $f \in \mathcal{F}^*$ and $0 < s \le s_f$, let $\mathcal{C}_{f,s}$ be a maximal solution of (3.1) with $c(0) = \epsilon_f s$ and $\vartheta_c = \frac{1}{2}\pi$ for some $c \in \mathcal{C}_{f,s}$. For instance, $[\mathcal{C}_{f,s_f}]$ simply is a circle with radius s_f centered at 0. Note that $s \mapsto [\mathcal{C}_{f,s}]$ is one-to-one, and clearly $[\mathcal{C}_{f,s}]$ is *not* a circle when $s < s_f$. The following properties of the curve $\mathcal{C}_{f,s}$ are immediate consequences of Lemma 3.8 and Theorem 3.13.

Proposition 4.4 *Let $f \in \mathcal{F}^*$ and $0 < s \le s_f$.*

(i) $\mathcal{C}_{f,s}$ is closed if and only if $\omega_f(\epsilon_f s) \in \mathbb{Q}$.
(ii) $\mathcal{C}_{f,s}$ is simple closed if and only if either $s = s_f$, or else $s < s_f$ and $\omega_f(\epsilon_f s) = 1/n$ for some $n \in \mathbb{N} \setminus \{1\}$.

It is clear that, as mentioned earlier and modulo rotations, the family $(\mathcal{C}_{f,s})_{0<s\le s_f}$ contains all closed maximal solutions that are untwisted, so in particular all Jordan solutions.

Proposition 4.5 *Let $f \in \mathcal{F}^*$, and assume that $\mathcal{C}$ is a closed maximal solution of (3.1). If $\mathcal{C}$ is untwisted then $[\mathcal{C}] = e^{i\vartheta}[\mathcal{C}_{f,s}]$ for some $\vartheta \in \mathbb{R}$ and a unique $0 < s \le s_f$.*

Combining Propositions 4.4 and 4.5, it is a simple task to find, at least formally, all Jordan solutions. For convenience, let

$$\mathbb{O}_f = \bigcup\nolimits_{n\ge 2} \left\{ 0 < s < s_f : \omega_f(\epsilon_f s) = \frac{1}{n} \right\}.$$

Though a next-to-trivial consequence of the above, the following theorem may nevertheless be regarded the main result of this section as it completely describes all Jordan solutions of (3.1) when $f \in \mathcal{F}^*$, and hence in fact even when merely $f \in \mathcal{F}$.

Theorem 4.6 *Let $f \in \mathcal{F}^*$, and assume that $\mathcal{C}$ is a closed maximal solution of* (3.1). *Then $\mathcal{C}$ is oriented counter-clockwise when $f > 0$, and clockwise when $f < 0$. Moreover, the following are equivalent:*

(i) $\mathcal{C}$ is a Jordan solution;
(ii) $[\mathcal{C}]$ either is a circle with radius s_f centered at 0, or else $[\mathcal{C}] = e^{i\vartheta}[\mathcal{C}_{f,s}]$ for some $\vartheta \in \mathbb{R}$ and a unique $s \in \mathbb{O}_f$.

Proof Recall that either $f > 0$ or $f < 0$ since $f \in \mathcal{F}^*$, and correspondingly $\epsilon_f = 1$ or $\epsilon_f = -1$. Clearly, $\mathcal{C}$ is oriented counter-clockwise in the former case, and clockwise in the latter [27, Ch. 1]. That (i)$\Leftrightarrow$(ii) is immediate from Theorem 3.13 together with Propositions 4.4 and 4.5. □

For every $f \in \mathcal{F}^*$, Theorem 4.6 establishes a bijection between $\mathbb{O}_f \subset\ ]0, s_f[$ and the non-circular Jordan solutions of (3.1) modulo rotations. Thus, to find all such solutions one only has to determine the set $\mathbb{O}_f$, most basically its cardinality. The remainder of this section aims at determining $\#\mathbb{O}_f$, on the one hand by establishing a practicable lower bound for ω_f, and on the other hand by devising a condition that ensures ω_f is monotone. For both tasks, it is convenient to assume henceforth that $F(s_f) = 0$ for every $f \in \mathcal{F}^*$. (The function F has been determined only up to an additive constant so far.) Note that $|f| \in \mathcal{F}^*$ whenever $f \in \mathcal{F}^*$, and $s_{|f|} = s_f$. For convenience, therefore, assume that $f > 0$ from now on. (If $f < 0$ then simply replace f by $-f$ in all that follows.) Then F is non-negative and (strictly) convex. Let $F_k = F^{(k)}(s_f)$ for $k \in \mathbb{N}_0$, so in particular $F_0 = F_1 = 0$ and $F_2 > 0$. With a view towards Theorem 4.6, for the further analysis it is helpful to derive an explicit formula for ω_f on $]0, s_f[$ as follows: For every $0 < s < s_f$ there exists a unique $s^* > s_f$ such that $F(s^*) = F(s)$, or more geometrically, s, s^* are the two intersection points of $\Phi_f(\mathbb{R}, s)$ with the real axis. Note that $s \mapsto s^*$ is smooth and decreasing, with $0^*_f := \lim_{s\to 0} s^* < \infty$ and $\lim_{s\to s_f} s^* = s_f$. With this,

$$\omega_f(s) = \frac{1}{\pi}\int_s^{s^*} \frac{u f(u)\,\mathrm{d}u}{\sqrt{F(s) - F(u)}\sqrt{2u + F(u) - F(s)}} \quad \forall 0 < s < s_f. \tag{4.1}$$

Utilizing (4.1), the proof of the following is a straightforward calculus exercise.

Proposition 4.7 *Let $f \in \mathcal{F}^*$. Then ω_f is smooth and positive on $]0, s_f[$, with*

$$\begin{aligned} \omega_f(s_f-) \quad &= \tfrac{1}{\sqrt{s_f F_2}}, \quad \omega_f'(s_f-) = 0, \\ \omega_f''(s_f-) &= \tfrac{9F_2^2 - 3s_f F_2(3F_2^2 - 2F_3) - s_f^2(3F_2F_4 - 5F_3^2)}{24\sqrt{s_f F_2}^5}. \end{aligned}$$

Since F is non-negative and convex, with $F(s_f) = F'(s_f) = 0$, the ratios $F''F/(F')^2$ and $(F')^2/F$ define positive smooth functions on $\mathbb{R}^+$, with their value for $s = s_f$ equal to $\frac{1}{2}$ and $2F_2$, respectively. The following simple observations are useful when establishing a lower bound for ω_f.

Proposition 4.8 *Let $f \in \mathcal{F}^*$, and assume that $F''F/(F')^2$ is increasing (respectively, decreasing) on $\mathbb{R}^+$. Then:*

(i) $s \mapsto s^*$ is concave (respectively, convex) on $]0, s_f[$;
(ii) $(F')^2/F$ is increasing (respectively, decreasing) on $\mathbb{R}^+$;
(iii) F' is convex (respectively, concave) on $\mathbb{R}^+$.

Using Proposition 4.8, it is now possible to establish a reasonably tight lower bound for ω_f on $]0, s_f[$, provided that f is *increasing*.

Lemma 4.9 *Let $f \in \mathcal{F}^*$ be increasing. If $F''F/(F')^2$ is increasing on $\mathbb{R}^+$ then*

$$\omega_f(s) > \frac{\sqrt{s_f}}{\pi}\sqrt{\frac{s^* - s_f}{F'(s^*)}}\int_0^{\pi} f\left(\sqrt{\frac{(s^*)^2 + s^2}{2} - \frac{(s^*)^2 - s^2}{2}\cos u}\right)\mathrm{d}u \quad \forall 0 < s < s_f. \tag{4.2}$$

If $F''F/(F')^2$ is decreasing on $\mathbb{R}^+$ then (4.2) holds with $\dfrac{s^* - s_f}{F'(s^*)}$ replaced by $\dfrac{s - s_f}{F'(s)}$.

Proof Assume for the time being that $F''F/(F')^2$ is increasing. It will first be shown that

$$-\frac{F'(s)}{s^* - s} \le \frac{F(s) - F(u)}{(s^* - u)(u - s)} \le \frac{F'(s^*)}{s^* - s} \quad \forall s < u < s^*. \tag{4.3}$$

To establish the left inequality in (4.3), notice that $g : [s, s^*] \to \mathbb{R}$ given by

$$g(u) = F(s) - F(u) + \frac{F'(s)}{s^* - s}(s^* - u)(u - s) \quad \forall s \le u \le s^*$$

satisfies $g(s) = g(s^*) = 0$, $g'(s) = 0$, and

$$g'(s^*) = -F'(s^*) - F'(s) = \sqrt{F(s)}\left(\sqrt{\frac{F'(s)^2}{F(s)}} - \sqrt{\frac{F'(s^*)^2}{F(s^*)}}\right) \le 0,$$

by Proposition 4.8(ii); also, by (iii) the function g' is concave. Consequently, if $g'(u_0) = 0$ for some $s < u_0 < s^*$ then $g'(u) \ge 0$ for all $s \le u \le u_0$, and $g(u_0) \ge 0$. Thus $g(u) \ge 0$ for all $s \le u \le s^*$, which proves the left inequality in (4.3). A completely analogous argument establishes the right inequality.

Next it will be shown that

$$\frac{(s^* + u)(u + s)}{2u + F(u) - F(s)} \ge s^* + s \quad \forall s \le u \le s^*. \tag{4.4}$$

To see (4.4), similarly to before observe that $h : [s, s^*] \to \mathbb{R}$ given by

$$h(u) = 2u + F(u) - F(s) - \frac{(s^* + u)(u + s)}{s + s^*} \quad \forall s \le u \le s^*$$

satisfies $h(s) = h(s^*) = 0$, and h' is convex. Since f is increasing, $h'(u_0) = 0$ for a unique $s \le u_0 \le s^*$, and $h(u_0)$ is a *minimal* value ≤ 0. In other words, $h(u) \le 0$ for all $s \le u \le s^*$, i.e., (4.4) holds.

Lastly, deduce from differentiating $F(s^*) = F(s)$ twice that $\mathrm{d}s^*/\mathrm{d}s|_{s=s_f} = -1$, and hence $s^* \le 2s_f - s$ by Proposition 4.8(i), so

$$(s^*)^2 - s^2 \ge (s^*)^2 - (2s_f - s^*)^2 = 4s_f(s^* - s_f). \tag{4.5}$$

With these preparations, for every $0 < s < s_f$ deduce from (4.1), together with (4.3), (4.4), and (4.5) that

$$\begin{aligned}
\omega_f(s) &= \frac{1}{\pi}\int_s^{s^*} \frac{uf(u)}{\sqrt{(s^* - u)(u - s)}\sqrt{(s^* + u)(u + s)}}\sqrt{\frac{(s^* - u)(u - s)}{F(s) - F(u)}}\sqrt{\frac{(s^* + u)(u + s)}{2u + F(u) - F(s)}}\,\mathrm{d}u \\
&\ge \frac{1}{\pi}\sqrt{\frac{s^* - s}{F'(s^*)}}\sqrt{s^* + s}\int_s^{s^*} \frac{uf(u)\,\mathrm{d}u}{\sqrt{\left((s^*)^2 - u^2\right)\left(u^2 - s^2\right)}} \\
&> \frac{1}{\pi}\sqrt{\frac{s^* - s_f}{F'(s^*)}}\int_0^{\pi} f\left(\sqrt{\frac{(s^*)^2 + s^2}{2} - \frac{(s^*)^2 - s^2}{2}\cos u}\right)\mathrm{d}u,
\end{aligned}$$

which is precisely (4.2). Finally, if $F''F/(F')^2$ is decreasing then it is readily checked that (4.3) holds with both inequalities reversed, whereas (4.4) remains valid unchanged, and $s^* \ge 2s_f - s$ since $s \mapsto s^*$ is convex, so (4.5) now reads

$$(s^*)^2 - s^2 \ge (2s_f - s)^2 - s^2 = 4s_f(s_f - s).$$

This shows that (4.2) remains valid, provided that $\dfrac{s^* - s_f}{F'(s^*)}$ is replaced by $\dfrac{s - s_f}{F'(s)}$. $\square$

Remark 4.10 (i) The right-hand side in (4.2) tends to $1/\sqrt{s_f F_2}$ as $s \to s_f$. Thus the lower bound in Lemma 4.9 is sharp at the right end of $]0, s_f[$. Moreover, equality holds in (4.2) for every s in case f is constant — although, strictly speaking, the lemma does not apply in this case because $f \notin \mathcal{F}$.

(ii) A variant of Lemma 4.9 holds for *decreasing* $f \in \mathcal{F}^*$ as well: While (4.3) remains valid in this case also, the right-hand side in (4.4) has to be replaced by the trivial lower bound $\frac{1}{2}(s^* + s)$. As a consequence, the right-hand side in (4.2) has to be divided by $\sqrt{2}$, resulting in a lower bound for $\omega_f(s)$ that typically is not sharp anywhere.

This section concludes with a discussion of the monotonicity of ω_f. Clearly, if one assumes ω_f to be, say, decreasing on $]0, s_f[$, then Theorem 4.6 together with Proposition 4.7 immediately yields the cardinality of $\mathbb{O}_f$, and thus the number of non-circular Jordan solutions of (3.1).

Theorem 4.11 *Let $f \in \mathcal{F}^*$, and assume that ω_f is decreasing on $]0, s_f[$. Then (3.1) has precisely $\#\Big(\big(\mathbb{N} \setminus \{1\}\big) \cap \big]1/\omega_f(0+), \sqrt{s_f F_2}\big[\Big)$ different non-circular Jordan solutions, modulo rotations.*

Proof Since ω_f is continuous and decreasing on $]0, s_f[$,

$$\{\omega_f(s) : 0 < s < s_f\} = \left]\frac{1}{\sqrt{s_f F_2}}, \omega_f(0+)\right[,$$

and hence $\#\mathbb{O}_f$ equals the number of integers $n \geq 2$ with $1/\omega_f(0+) < n < \sqrt{s_f F_2}$. □

To establish a condition that ensures ω_f is decreasing on $]0, s_f[$, notice that for every $f \in \mathcal{F}^*$ the ratio F/F' defines a smooth function on $\mathbb{R}^+$, with its value and derivative for $s = s_f$ equal to 0 and $\frac{1}{2}$, respectively.

Lemma 4.12 *Let $f \in \mathcal{F}^*$, and assume there exists $a \in \mathbb{R}$ such that*

$$2\sqrt{s}\frac{\mathrm{d}}{\mathrm{d}s}\left(\sqrt{s}f(s)\frac{F(s)}{F'(s)}\right) - sf(s) + \frac{f(s)F(s)}{4} \geq aF'(s)\sqrt{s} \quad \forall 0 < s < 0_f^*. \tag{4.6}$$

Then $\omega_f'(s) < 0$ for all $0 < s < s_f$.

Remark 4.13 The left- and right-hand sides in (4.6) both vanish for $s = s_f$, with derivatives equal to $F_2 - 1/(2s_f) - F_3/(3F_2)$ and $aF_2\sqrt{s_f}$, respectively. Thus, if a as in Lemma 4.12 exists at all then necessarily $a = (s_f(F_2^2 - \frac{1}{3}F_3) - \frac{1}{2}F_2)/(s_f^{3/2}F_2^2)$.

The proof of Lemma 4.12 makes use of a simple tailor-made calculus fact the verification of which once more is left to the interested reader; see also [13,Thm.2.1].

Proposition 4.14 *Let $f \in \mathcal{F}^*$, and assume $g : \mathbb{R}^+ \to \mathbb{R}$ is smooth. Then, for every $0 < s < s_f$,*

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_s^{s^*}\frac{g(u)\,\mathrm{d}u}{\sqrt{F(s) - F(u)}} = \frac{F'(s)}{F(s)}\int_s^{s^*}\frac{1}{\sqrt{F(s) - F(u)}}\left(\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{g(u)F(u)}{F'(u)}\right) - \frac{g(u)}{2}\right)\mathrm{d}u.$$

Proof of Lemma 4.12 For every $0 < s < s_f$,

$$0 \leq \frac{F(s) - F(u)}{2u} \leq 1 - \frac{s}{s^*} < 1 \quad \forall s < u < s^*, \tag{4.7}$$

and consequently, with (4.1) and the binomial formula,

$$
\begin{aligned}
\omega_f(s) &= \frac{1}{\pi\sqrt{2}} \int_s^{s^*} \frac{\sqrt{u}\, f(u)}{\sqrt{F(s)-F(u)}} \left(1 - \frac{F(s)-F(u)}{2u}\right)^{-1/2} \mathrm{d}u \\
&= \frac{1}{\pi\sqrt{2}} \int_s^{s^*} \sum\nolimits_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n 2^{-n} u^{1/2-n} f(u)\big(F(s)-F(u)\big)^{n-1/2} \mathrm{d}u \\
&= \frac{1}{\pi\sqrt{2}} \sum\nolimits_{n=0}^{\infty} \binom{2n}{n} 2^{-3n} g_n(s) \\
&= \frac{1}{\pi\sqrt{2}} \left(g_0(s) + \frac{g_1(s)}{4} + \sum\nolimits_{n=2}^{\infty} \binom{2n}{n} 2^{-3n} g_n(s) \right),
\end{aligned}
$$

where the second-to-last equality is due to uniform convergence, and for every integer $n \geq 0$,

$$
g_n(s) = \int_s^{s^*} u^{1/2-n} f(u)\big(F(s)-F(u)\big)^{n-1/2} \mathrm{d}u \quad \forall 0 < s < s_f.
$$

The derivative of g_0 can be computed using Proposition 4.14, for every $0 < s < s_f$,

$$
g_0'(s) = \frac{F'(s)}{2F(s)} \int_s^{s^*} \frac{1}{\sqrt{u\big(F(s)-F(u)\big)}} \left(2\sqrt{u} \frac{\mathrm{d}}{\mathrm{d}u} \left(\sqrt{u}\, f(u) \frac{F(u)}{F'(u)} \right) - uf(u) \right) \mathrm{d}u,
$$

whereas for $n \geq 1$, the derivative of g_n simply is obtained by formal differentiation,

$$
g_n'(s) = \big(n - \tfrac{1}{2}\big) F'(s) \int_s^{s^*} u^{1/2-n} f(u)\big(F(s)-F(u)\big)^{n-3/2} \mathrm{d}u \quad \forall 0 < s < s_f.
$$

Note that $g_n' < 0$ for all $n \geq 1$ since $F' < 0$ on $]0, s_f[$. By means of (4.7), it is easily seen that the termwise differentiated series $\sum_{n=2}^{\infty} \binom{2n}{n} 2^{-3n} g_n'(s)$ converges locally uniformly on $]0, s_f[$, and hence

$$
\begin{aligned}
\omega_f'(s) &= \frac{1}{\pi\sqrt{2}} \left(g_0'(s) + \frac{g_1(s)'}{4} + \sum\nolimits_{n=2}^{\infty} \binom{2n}{n} 2^{-3n} g_n'(s) \right) \\
&< \frac{1}{\pi\sqrt{2}} \left(g_0'(s) + \frac{g_1'(s)}{4} \right) \\
&= \frac{F'(s)}{2\pi\sqrt{2}F(s)} \int_s^{s^*} \frac{1}{\sqrt{u\big(F(s)-F(u)\big)}} \\
&\quad \left(2\sqrt{u} \frac{\mathrm{d}}{\mathrm{d}u} \left(\sqrt{u}\, f(u) \frac{F(u)}{F'(u)} \right) - uf(u) + \frac{f(u)F(s)}{4} \right) \mathrm{d}u \\
&\leq \frac{aF'(s)}{2\pi\sqrt{2}F(s)} \int_s^{s^*} \frac{F'(u)\, \mathrm{d}u}{\sqrt{F(s)-F(u)}} = 0,
\end{aligned}
$$

where the last inequality is a consequence of (4.6) and (4.7). □

Remark 4.15 With a view towards Proposition 4.14 and the proof of Lemma 4.12, it is tempting to write down an exact formula (rather than an upper bound) for $\omega_f'(s)$, namely

$$\omega_f'(s) = \frac{F'(s)}{2\pi\sqrt{2}F(s)} \int_s^{s^*} \frac{1}{\sqrt{u\big(F(s)-F(u)\big)}} \left(2\sqrt{u}\frac{\mathrm{d}}{\mathrm{d}u}\left(\sqrt{u}f(u)\frac{F(u)}{F'(u)}\right) - uf(u) \right. \tag{4.8}$$

$$\left. + \frac{f(u)F(s)}{4}\psi\left(\frac{F(s)-F(u)}{2u}\right)\right) \mathrm{d}u,$$

with the (real-analytic) function

$$\psi(t) = \frac{2t}{\sqrt{1-t}^3} + \frac{2}{1+\sqrt{1-t}} \quad \forall t < 1.$$

Note that ψ is convex on $[0, 1[$, with $\psi(0) = 1$ and $\psi'(0) = \frac{9}{4}$. Due to its "non-local" nature, (4.8) appears to be rather unwieldy. In particular, the integrand typically changes sign in $]s, s^*[$. As a consequence, any general statement about the sign of ω_f' is bound to be a delicate affair, a fact for which the next section is going to provide ample evidence.

5 An Example: The Monomial Family

This final section applies the results of the preceding sections to the monomials $f_b(s) = s^b$, with $b \in \mathbb{R}$. Naturally, the analysis is quite specific to that particular family of functions, though the techniques applied here likely are useful also when dealing with other classes of functions. Moreover, the section illustrates how applying the results of the present article, notably Theorem 4.11, though quite trivial in theory, may nonetheless pose a considerable challenge in practice. This is not an uncommon situation: The reader likely is familiar with similar, seemingly simple problems in non-linear analysis that also require for their resolution lengthy, potentially delicate and unenlightening computations; see, for instance, [1, 9, 29, 37].

Recall from the previous section that $f_b \in \mathcal{F}$ for every $b \in \mathbb{R} \setminus \{-1, 0\}$, and first consider the case $b < -1$, where $\epsilon_{f_b} = -1$ and $\mathbb{I}_{f_b} = -\mathbb{R}^+$. Consequently, Proposition 4.3 shows that the only Jordan solution of $\kappa = r^b$ is the (counter-clockwise oriented) unit circle. Next, the case $b = -1$ has been considered already in Example 2.12(ii): Every (counter-clockwise oriented) circle centered at 0 is a Jordan solution of $\kappa = 1/r$, and there are no other maximal solutions that are bounded, let alone closed or Jordan. It remains to consider the case $b > -1$, where $f_b \in \mathcal{F}^*$ unless $b = 0$. In this case, $\epsilon_{f_b} = 1$ and $\mathbb{I}_{f_b} = \mathbb{R}^+$, so Theorem 4.6 applies. Moreover, $s_{f_b} = 1$, $0^*_{f_b} = (b+2)^{1/(b+1)}$, and

$$F(s) = \frac{s^{b+2} - (b+2)s + b + 1}{b+2} \quad \forall s \in \mathbb{R}^+,$$

as well as $F_k = \prod_{\ell=2}^{k}(b + 3 - \ell)$ for all $k \geq 2$. By Proposition 4.7, ω_{f_b} is a smooth positive function on $]0, 1[$ with

$$\omega_{f_b}(1-) = \frac{1}{\sqrt{b+1}}, \quad \omega_{f_b}'(1-) = 0, \quad \omega_{f_b}''(1-) = \frac{b^2}{12\sqrt{b+1}}. \tag{5.1}$$

Thus if $-1 < b < 0$ then $\omega_{f_b}(s) > 1$ for all $0 < s < 1$, by Lemma 2.9, hence $\mathbb{O}_{f_b} = \varnothing$, and again the only Jordan solution of $\kappa = r^b$ is the (counter-clockwise oriented) unit circle.

By contrast, if $b = 0$ then $\omega_{f_0}(s) = 1$ for all $0 < s < 1$. Though (3.7) fails in this case, as $f_0 \notin \mathcal{F}$, it is clear that every (counter-clockwise oriented) circle with radius 1 is a Jordan solution, and there are no other maximal solutions whatsoever. Correspondingly, $s^* = 2 - s$ and

$$\begin{aligned}\omega_{f_0}(s) &= \frac{2}{\pi}\int_s^{2-s} \frac{u\,\mathrm{d}u}{\sqrt{u^2 - s^2}\sqrt{(2-s)^2 - u^2}} \\ &= \frac{1}{\pi}\int_{s^2}^{(2-s)^2} \frac{\mathrm{d}u}{\sqrt{u - s^2}\sqrt{(2-s)^2 - u}} = 1 \quad \forall 0 < s < 1.\end{aligned}$$

The only case yet to be considered, therefore, is $b > 0$. In this case,

$$\begin{aligned}\omega_{f_b}(0+) &= \frac{1}{\pi}\int_0^{0^*_{f_b}} \frac{u^{b+1}\mathrm{d}u}{\sqrt{F(0) - F(u)}\sqrt{2u + F(u) - F(0)}} \\ &= \frac{1}{\pi}\int_0^{(b+2)^{1/(b+1)}} \frac{u^{b+1}\mathrm{d}u}{\sqrt{u^2 - u^{2b+4}/(b+1)^2}} = \frac{1}{2} + \frac{1}{2(b+1)},\end{aligned} \tag{5.2}$$

and consequently

$$\omega_{f_b}(0+) - \omega_{f_b}(1-) = \frac{b^2}{2(b+1)(b+2+2\sqrt{b+1})} > 0.$$

Since ω_{f_b} attains a non-degenerate local minimum as $s \to 1$ by (5.1), the following certainly is a plausible speculation.

Conjecture 5.1 *For every $b > 0$ the function ω_{f_b} is decreasing on $]0, 1[$.*

At the time of this writing, the author has been able to establish the correctness of this conjecture only for $b \geq \frac{3}{2}$; see Lemma 5.3 below. For smaller b, a somewhat weaker substitute is presented. Concretely, observe that $\frac{1}{2} < \omega_{f_b}(0+) < 1$ for all $b > 0$, and if $0 < b < 3$ then also $\frac{1}{2} < \omega_{f_b}(1-) < 1$. If b is not too large then these bounds are valid for all intermediate values as well.

Lemma 5.2 *If $0 < b \leq \frac{3}{2}$ then $\frac{1}{2} < \omega_{f_b}(s) < 1$ for all $0 < s < 1$.*

The proof of Lemma 5.2 presented below makes use of several inequalities, two of which may be of independent interest: On the one hand, elementary calculus shows that

$$a \leq \left(\frac{(a+1)^{b+1} - 1}{a}\right)^{1/b} - (b+1)^{1/b} \leq a\frac{(b+1)^{1/b}}{2} \quad \forall a \in \mathbb{R}^+, 0 < b \leq 1,$$

whereas for $b \geq 1$ both inequalities are reversed. In other words, since $\max\left\{\frac{1}{2}(b+1)^{1/b}, 1\right\}$ equals $\frac{1}{2}(b+1)^{1/b}$ if $0 < b \leq 1$, and equals 1 if $b \geq 1$,

$$\begin{aligned}a\min\left\{\frac{(b+1)^{1/b}}{2}, 1\right\} &\leq \left(\frac{(a+1)^{b+1} - 1}{a}\right)^{1/b} \\ -(b+1)^{1/b} &\leq a\max\left\{\frac{(b+1)^{1/b}}{2}, 1\right\} \quad \forall a, b \in \mathbb{R}^+.\end{aligned} \tag{5.3}$$

On the other hand, as a special case of an optimal Gautschi inequality established in [21],

$$a + \frac{1}{4} < \frac{\Gamma(a+1)^2}{\Gamma(a+1/2)^2} < a + \frac{1}{\pi} \quad \forall a \in \mathbb{R}^+, \tag{5.4}$$

and both bounds are sharp, as the left (respectively, right) inequality becomes an equality as $a \to \infty$ (respectively, $a \to 0$).

Proof of Lemma 5.2 Since $0 < \omega_{f_b}(s) < 1$ for all $0 < s < 1$ whenever $b > 0$, by Lemma 2.9(iii), it only needs to be shown that $\omega_{f_b}(s) > \frac{1}{2}$. Clearly, f_b is increasing, and it is readily checked that $F''F/(F')^2$ is increasing on $\mathbb{R}^+$ as well. By (4.2),

$$\omega_{f_b}(s) \geq \frac{1}{2^{b/2}\pi}\sqrt{\frac{s^*-1}{(s^*)^{b+1}-1}}\big((s^*)^2+s^2\big)^{b/2}\int_0^\pi \left(1-\frac{(s^*)^2-s^2}{(s^*)^2+s^2}\cos u\right)^{b/2} \mathrm{d}u \quad \forall 0<s<1,$$

and utilizing the elementary estimate

$$\int_0^\pi (1-a\cos u)^b \mathrm{d}u \geq \frac{2^b\sqrt{\pi}\,\Gamma(b+1/2)}{\Gamma(b+1)} \quad \forall 0 \leq a, b \leq 1,$$

it follows that, for every $0 < b \leq 2$,

$$\omega_{f_b}(s)^2 \geq \frac{1}{\pi}\cdot\frac{s^*-1}{(s^*)^{b+1}-1}\big((s^*)^2+s^2\big)^b\frac{\Gamma\big((b+1)/2\big)^2}{\Gamma\big((b+2)/2\big)^2} \quad \forall 0<s<1.$$

Recall that $s^* > 1$ for every $0 < s < 1$. Thus, applying (5.3) and (5.4) yields the lower bound, valid whenever $0 < b \leq 2$,

$$\omega_{f_b}(s)^2 > \frac{\big((s^*)^2+s^2\big)^b}{\big((b+1)^{1/b}+(s^*-1)\max\{(b+1)^{1/b}/2, 1\}\big)^b}\cdot\frac{2}{\pi b+2} \quad \forall 0<s<1. \tag{5.5}$$

Also, recall from Proposition 4.8(i) that $s \mapsto s^*$ is concave, with $s^*|_{s=0} = 0^*_{f_b} = (b+2)^{1/(b+1)}$ and $s^*|_{s=1} = 1$, and hence

$$s^* \geq (b+2)^{1/(b+1)}(1-s)+s \quad \forall 0<s<1. \tag{5.6}$$

Replacing s on the right in (5.5) by the lower bound in terms of s^* provided by (5.6), and requiring that the resulting expression still be $> \frac{1}{4}$ for all $0 < s < 1$ is equivalent to requiring that

$$p(b, s^*-1) > 0 \quad \forall 1 < s^* < (b+2)^{1/(b+1)},$$

with the continuous function $p : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ given by

$$\begin{aligned} p(b,t) = (t+1)^2 + \left(\frac{t}{(b+2)^{1/(b+1)}-1}-1\right)^2 \\ - \left(\frac{\pi b+2}{8}\right)^{1/b}\left((b+1)^{1/b}+t\max\left\{\frac{(b+1)^{1/b}}{2}, 1\right\}\right). \end{aligned}$$

Thus the assertion of the lemma immediately follows, as soon as it is shown that in fact

$$p(b,t) > 0 \quad \forall 0 < b \leq \frac{3}{2}, t \in \mathbb{R}. \tag{5.7}$$

In other words, to prove the lemma it suffices to establish (5.7), and this will now be done. To this end, notice that $p(b, \cdot)$ is a quadratic polynomial,

$$p(b,t) = p_2(b)t^2 - 2p_1(b)t + p_0(b) \quad \forall b > 0, t \in \mathbb{R},$$

with continuous, positive coefficients $p_0, p_1, p_2 : \mathbb{R}^+ \to \mathbb{R}^+$ given by

$$
\begin{aligned}
p_0(b) &= 2 - \left(\frac{\pi b + 2}{8}\right)^{1/b} (b+1)^{1/b}, \\
p_1(b) &= \frac{1}{(b+2)^{1/(b+1)} - 1} - 1 + \frac{1}{2}\left(\frac{\pi b + 2}{8}\right)^{1/b} \max\left\{\frac{(b+1)^{1/b}}{2}, 1\right\}, \\
p_2(b) &= \frac{1}{\left((b+2)^{1/(b+1)} - 1\right)^2} + 1,
\end{aligned}
$$

and hence (5.7) holds, provided that

$$
p_0(b)p_2(b) > p_1(b)^2 \quad \forall 0 < b \le \frac{3}{2}. \tag{5.8}
$$

Now, it is readily checked that p_0 and p_1 are decreasing and increasing on $]0, \frac{3}{2}]$, respectively, and hence to establish (5.8), it suffices to verify that

$$
\begin{aligned}
&\sqrt{p_0(1)}\sqrt{p_2(b)} > p_1(1) \quad \forall 0 < b \le 1 \quad \text{and} \\
&\sqrt{p_0\left(\frac{3}{2}\right)}\sqrt{p_2(b)} > p_1\left(\frac{3}{2}\right) \quad \forall 1 \le b \le \frac{3}{2}.
\end{aligned} \tag{5.9}
$$

Notice that

$$
\sqrt{p_2(b)} \ge \frac{1}{\sqrt{2}} \cdot \frac{(b+2)^{1/(b+1)}}{(b+2)^{1/(b+1)} - 1} \quad \forall b \in \mathbb{R}^+,
$$

and since the lower bound on the right is increasing in b, clearly (5.9) holds, provided that

$$
\sqrt{p_0(1)}\frac{2}{\sqrt{2}} > p_1(1) \quad \text{and} \quad \sqrt{p_0\left(\frac{3}{2}\right)}\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{\sqrt{3}-1} > p_1\left(\frac{3}{2}\right). \tag{5.10}
$$

Utilizing the rough (rational) estimates

$$
\begin{aligned}
p_0(1) &= \frac{6-\pi}{4} > \frac{2}{3}, \quad p_0\left(\frac{3}{2}\right) = 2 - \frac{1}{16}(40 + 30\pi)^{2/3} > \left(\frac{3}{5}\right)^2, \\
p_1(1) &= \frac{8\sqrt{3} - 6 + \pi}{16} < \frac{3}{4}, \quad p_1\left(\frac{3}{2}\right) = \frac{2^{7/5} - 7^{2/5}}{7^{2/5} - 2^{2/5}} + \frac{1}{32}(16 + 12\pi)^{2/3} < 1,
\end{aligned}
$$

it is readily seen that (5.10) indeed is correct. This proves (5.8), which in turn implies (5.7). As detailed earlier, the latter proves the lemma. □

As a consequence of Lemma 5.2, $\mathbb{O}_{f_b} = \varnothing$ also when $0 < b \le \frac{3}{2}$, and again the only Jordan solution of $\kappa = r^b$ is the (counter-clockwise oriented) unit circle.

Lastly consider the case $b \ge \frac{3}{2}$. In this case, the conclusion of Conjecture 5.1 definitely holds.

Lemma 5.3 *If $b \ge \frac{3}{2}$ then the function ω_{f_b} is decreasing on* $]0, 1[$.

The proof of Lemma 5.3 presented below makes use of a simple calculus fact: Given $n \in \mathbb{N}$, non-zero real numbers $a_1, \ldots, a_n$, and real numbers $b_1 > \ldots > b_n$, consider the real-analytic function $g : \mathbb{R} \to \mathbb{R}$ given by

$$
g(t) = \sum\nolimits_{\ell=1}^{n} a_\ell e^{b_\ell t}, \tag{5.11}
$$

and let $\sigma(g)$ be the number of sign changes in the finite sequence $(a_1, \ldots, a_n)$, more formally, $\sigma(g) = \#\{1 \le \ell \le n : a_{\ell-1}a_\ell < 0\}$ where $a_0 := a_1$. Plainly, $0 \le \sigma(g) \le n-1$. Since $\lim_{t\to-\infty} g(t)e^{-b_n t} = a_n \neq 0$ and $\lim_{t\to\infty} g(t)e^{-b_1 t} = a_1 \neq 0$, the equation $g(t) = 0$ has only finitely many real roots, each of which has finite multiplicity. The following variant of Descartes' rule [16] makes this more precise.

Proposition 5.4 *Let $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{R} \setminus \{0\}$, and $b_1, \ldots, b_n \in \mathbb{R}$ with $b_1 > \ldots > b_n$. Then the total number of real roots (counted with multiplicities) of $g(t) = 0$, with g as in (5.11), equals $\sigma(g) - 2k$ for some $k \in \mathbb{N}_0$.*

Proof of Lemma 5.3 Though computationally intense in its details, the following argument has a simple basic strategy: Intending to utilize Lemma 4.12 with $f = f_b$, notice first that (4.6), with $a = \frac{1}{6}(4b+3)/(b+1)$ as per Remark 4.13, can be written equivalently but more concisely as

$$p\left(2(b+1), \frac{\log s}{2}\right) \ge 0 \quad \forall 0 < s < (b+2)^{1/(b+1)},$$

with the real-analytic function $p : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\begin{aligned} p(\varepsilon, t) = {} & 6\varepsilon e^{(4\varepsilon-1)t} - 4(\varepsilon+2)(2\varepsilon-1)e^{3\varepsilon t} + 9\varepsilon(\varepsilon-2)e^{(3\varepsilon-1)t} + 3\varepsilon^2 t^{3(\varepsilon-1)t} \\ & + 12(\varepsilon+2)(2\varepsilon-1)e^{2\varepsilon t} - 6\varepsilon(5\varepsilon-3)e^{(2\varepsilon-1)t} - 18\varepsilon^2 e^{(2\varepsilon-3)t} \\ & - 12(\varepsilon+2)(2\varepsilon-1)e^{\varepsilon t} + 3\varepsilon(\varepsilon+2)(4\varepsilon-1)e^{(\varepsilon-1)t} \\ & - 3\varepsilon^2(4\varepsilon-5)e^{(\varepsilon-3)t} + 4(\varepsilon+2)(2\varepsilon-1). \end{aligned} \tag{5.12}$$

Since $b \ge \frac{3}{2}$ precisely if $\varepsilon = 2(b+1) \ge 5$, the assertion of the lemma immediately follows from Lemma 4.12, as soon as it is shown that in fact

$$p(\varepsilon, t) \ge 0 \quad \forall \varepsilon \ge 5, t \in \mathbb{R}. \tag{5.13}$$

Thus, to prove the lemma it suffices to establish (5.13), and this will now be done in several steps. Usage of the same symbols as in the proof of Lemma 5.2 will hopefully not confuse the reader but rather highlight the parallels between both proofs.

Henceforth assume $\varepsilon \ge 5$, and notice at the outset that $\sigma\big(p(\varepsilon, \cdot)\big) = 6$, as well as $\partial^k p/\partial t^k(\varepsilon, 0) = 0$ for $k = 0, 1, 2, 3$, whereas $\partial^4 p/\partial t^4(\varepsilon, 0) = 24\varepsilon^3(\varepsilon+2)(2\varepsilon^2 - 11\varepsilon + 8) > 0$. For every $\varepsilon > 0$, therefore, $p(\varepsilon, \cdot)$ has $t = 0$ as a 4-fold root, and so by Proposition 5.4 has either two or zero additional real roots. In the latter case, clearly (5.13) is correct.

First it will be shown that $p(\varepsilon, t) \ge 0$ for all $t \in \mathbb{R}$, provided that ε is large enough. To this end, consider the real-analytic function $\widehat{p} : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\begin{aligned} \widehat{p}(\varepsilon, t) = {} & 2\varepsilon e^{(4\varepsilon-1)t} - (2\varepsilon-1)(3\varepsilon-1)e^{3\varepsilon t} + 2\varepsilon(3\varepsilon-1)e^{(3\varepsilon-1)t} \\ & -2(3\varepsilon-1)e^{2\varepsilon t} + (\varepsilon-1)e^{\varepsilon t}. \end{aligned}$$

Notice that $\sigma\big(\widehat{p}(\varepsilon, \cdot)\big) = 4$, with $t = 0$ being a 4-fold root, and consequently $\widehat{p}(\varepsilon, t) \ge 0$ for all $t \in \mathbb{R}$, by Proposition 5.4. Thus

$$\begin{aligned} p(\varepsilon, t) & \ge p(\varepsilon, t) - 4\frac{\varepsilon+2}{3\varepsilon-1}\widehat{p}(\varepsilon, t) \\ & = 2\varepsilon\frac{5\varepsilon-11}{3\varepsilon-1}\varepsilon e^{(4\varepsilon-1)t} + 0 \cdot e^{3\varepsilon t} + \varepsilon(\varepsilon-34)e^{(3\varepsilon-1)t} + 3\varepsilon^2 e^{3(\varepsilon-1)t} + \ldots, \end{aligned} \tag{5.14}$$

where ... indicates the remaining terms of $p(\varepsilon, t)$ from (5.12), of which only the coefficients of $e^{2\varepsilon t}$ and $e^{\varepsilon t}$ differ from those in (5.12), both being larger now in absolute value but having kept their respective signs. Now, if $\varepsilon \geq 34$ then $\sigma\big(p(\varepsilon, \cdot) - 4(\varepsilon+2)/(3\varepsilon-1)\widehat{p}(\varepsilon, \cdot)\big) = 4$, and since $t = 0$ is a 4-fold root of both $p(\varepsilon, \cdot)$ and $\widehat{p}(\varepsilon, \cdot)$, the right-hand side of (5.14) is ≥ 0 for all $t \in \mathbb{R}$, by Proposition 5.4. In particular, $p(\varepsilon, t) \geq 0$ for all $\varepsilon \geq 34$ and $t \in \mathbb{R}$.

To consider the remaining cases in (5.13), for the remainder of this proof assume $5 \leq \varepsilon \leq 34$, and for computational convenience let $\varepsilon = \nu + 5$, so $0 \leq \nu \leq 29$. Notice that

$$p(\varepsilon, t) = \sum\nolimits_{n=0}^{\infty} \frac{\partial^n p}{\partial t^n}(\varepsilon, 0)\frac{t^n}{n!} = t^4 \sum\nolimits_{n=0}^{\infty} p_n(\nu)\frac{t^n}{(n+4)!},$$

where for every $n \in \mathbb{N}_0$,

$$\begin{aligned}
p_n(\nu) &= \frac{\partial^{n+4} p}{\partial t^{n+4}}(\nu+5, 0)\\
&= 6(\nu+5)(4\nu+19)^{n+4} - 4(\nu+7)(2\nu+9)(3\nu+15)^{n+4}\\
&\quad + 9(\nu+5)(\nu+3)(3\nu+14)^{n+4} + 3(\nu+5)^2(3\nu+12)^{n+4}\\
&\quad + 12(\nu+7)(2\nu+9)(2\nu+10)^{n+4}\\
&\quad - 6(\nu+5)(5\nu+22)(2\nu+9)^{n+4} - 18(\nu+5)^2(2\nu+7)^{n+4}\\
&\quad - 12(\nu+7)(2\nu+9)(\nu+5)^{n+4}\\
&\quad + 3(\nu+5)(\nu+7)(4\nu+19)(\nu+4)^{n+4} - 3(\nu+5)^2(4\nu+15)(\nu+2)^{n+4}
\end{aligned}$$

is a polynomial of degree $n + 6$; for example,

$$\begin{aligned}
p_0(\nu) &= 48\nu^6 + 1272\nu^5 + 13464\nu^4 + 71664\nu^3 + 195360\nu^2 + 235800\nu + 63000\\
&= 24(\nu+5)^3(\nu+7)(2\nu^2+9\nu+3),\\
p_1(\nu) &= 336\nu^7 + 10440\nu^6 + 135240\nu^5 + 939840\nu^4\\
&\quad + 3734784\nu^3 + 8270760\nu^2 + 8893800\nu + 2898000\\
&= 24(\nu+5)^3(\nu+7)(14\nu^3 + 127\nu^2 + 321\nu + 138).
\end{aligned}$$

To establish (5.13) for all $5 \leq \varepsilon \leq 34$ and $t \geq 0$, clearly it is enough to demonstrate that

$$p_n(\nu) \geq 0 \quad \forall 0 \leq \nu \leq 29, n \in \mathbb{N}_0. \tag{5.15}$$

In order to do this, notice that for all $0 \leq \nu \leq 29$ the three inequalities

$$\begin{aligned}
3(\nu+5)(4\nu+19)^{n+4} &\geq 2(\nu+7)(2\nu+9)(3\nu+15)^{n+4},\\
3(\nu+3)(3\nu+14)^{n+4} &\geq 2(5\nu+22)(2\nu+9)^{n+4},\\
(3\nu+12)^{n+4} &\geq 6(2\nu+7)^{n+4},
\end{aligned}$$

hold simultaneously, and hence $p_n(\nu) \geq 0$, provided that $n \geq 13$. Thus (5.15) is correct for all $n \geq 13$. For the remaining cases $2 \leq n \leq 12$, a straightforward albeit tedious calculation (involving only integers, and much aided by symbolic mathematical software) shows that, just as for p_0 and p_1, all coefficients of p_n are positive (integers). Hence $p_n(\nu) \geq 0$ for all $0 \leq \nu \leq 29$ and $n \in \mathbb{N}_0$, i.e., (5.15) indeed is correct. As seen earlier, this yields $p(\varepsilon, t) \geq 0$ for all $5 \leq \varepsilon \leq 34$ and $t \geq 0$.

Finally, it remains to establish (5.13) for $5 \leq \varepsilon \leq 34$ and $t < 0$. To this end, consider the real-analytic function $q : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\begin{aligned} q(\varepsilon, t) = \; & e^{(4\varepsilon-1)t} p(\varepsilon, -t) \\ = \; & 4(\varepsilon+2)(2\varepsilon-1)e^{(4\varepsilon-1)t} - 3\varepsilon^2(4\varepsilon-5)e^{(3\varepsilon+2)t} + 3\varepsilon(\varepsilon+2)(4\varepsilon-1)e^{3\varepsilon t} \\ & - 12(\varepsilon+2)(2\varepsilon-1)e^{(3\varepsilon-1)t} - 18\varepsilon^2 e^{2(\varepsilon+1)t} \\ & - 6\varepsilon(5\varepsilon-3)e^{2\varepsilon t} + 12(\varepsilon+2)(2\varepsilon-1)e^{(2\varepsilon-1)t} \\ & + 3\varepsilon^2 t^{(\varepsilon+2)t} + 9\varepsilon(\varepsilon-2)e^{\varepsilon t} - 4(\varepsilon+2)(2\varepsilon-1)e^{(\varepsilon-1)t} + 6\varepsilon, \end{aligned}$$

and observe that the validity of $p(\varepsilon, t) \geq 0$ for $5 \leq \varepsilon \leq 34$ and $t < 0$ follows from

$$q(\varepsilon, t) \geq 0 \quad \forall 5 \leq \varepsilon \leq 34, t \geq 0, \tag{5.16}$$

so it is sufficient to establish (5.16). To do this, it is natural to imitate the earlier argument: Write

$$q(\varepsilon, t) = \sum\nolimits_{n=0}^{\infty} \frac{\partial^n q}{\partial t^n}(\varepsilon, 0)\frac{t^n}{n!} = t^4 \sum\nolimits_{n=0}^{\infty} q_n(\nu)\frac{t^n}{(n+4)!},$$

where for every $n \in \mathbb{N}_0$,

$$\begin{aligned} q_n(\nu) &= \frac{\partial^{n+4} q}{\partial t^{n+4}}(\nu+5, 0) \\ &= 4(\nu+7)(2\nu+9)(4\nu+19)^{n+4} - 3(\nu+5)^2(4\nu+15)(3\nu+17)^{n+4} \\ &\quad + (\nu+7)(4\nu+19)(3\nu+15)^{n+5} - 12(\nu+7)(2\nu+9)(3\nu+14)^{n+4} \\ &\quad - 18(\nu+5)^2(2\nu+12)^{n+4} - 3(5\nu+22)(2\nu+10)^{n+5} + 12(\nu+7)(2\nu+9)^{n+5} \\ &\quad + 3(\nu+5)^2(\nu+7)^{n+4} + 9(\nu+3)(\nu+5)^{n+5} - 4(\nu+7)(2\nu+9)(\nu+4)^{n+4} \end{aligned}$$

again is a polynomial of degree $n+6$; for example,

$$\begin{aligned} q_0(\nu) &= p_0(\nu) = 24(\nu+5)^3(\nu+7)(2\nu^2+9\nu+3), \\ q_1(\nu) &= 624\nu^7 + 19560\nu^6 + 254880\nu^5 + 1772520\nu^4 + 6980496\nu^3 + 15004440\nu^2 \\ &\quad + 14767200\nu + 3087000 \\ &= 24(\nu+5)^3(\nu+7)(26\nu^3+243\nu^2+594\nu+147). \end{aligned}$$

In complete analogy to (5.15), it suffices to show that

$$q_n(\nu) \geq 0 \quad \forall 0 \leq \nu \leq 29, n \in \mathbb{N}_0. \tag{5.17}$$

For all $0 \leq \nu \leq 29$, it is readily checked that

$$\begin{aligned} (4\nu+19)^{n+4} \geq \max\Bigg\{ & \frac{3(\nu+5)^2(4\nu+15)}{(\nu+7)(2\nu+9)}(3\nu+17)^{n+4}, 12(3\nu+14)^{n+4}, \\ & \frac{18(\nu+5)^2}{(\nu+7)(2\nu+9)}(2\nu+12)^{n+4}, \frac{3(5\nu+22)}{(\nu+7)(2\nu+9)}(2\nu+10)^{n+5} \Bigg\}, \end{aligned}$$

and hence $q_n(\nu) \geq 0$, provided that $n \geq 44$. Similarly to before, it can be confirmed by direct calculation that for $q_2, \ldots, q_{43}$ all coefficients are positive (integers), just as for q_0 and q_1. In other words, (5.17) is correct, which in turn yields $p(\varepsilon, t) \geq 0$ for all $5 \leq \varepsilon \leq 34$ and $t \leq 0$. At long last, therefore, (5.13) has been established. As detailed earlier, an application of Lemma 4.12 now completes the proof. □

Remark 5.5 (i) With the same symbols as in the proof of Lemma 5.3, for every $n \in \mathbb{N}_0$ let ν_n be the largest real root of $p_n q_n$. In essence, the above proof hinges on the fact that $\nu_n < 0$ for every n. A careful analysis reveals that $\sup_{n\in\mathbb{N}_0} \nu_n = \nu_1 = -0.2781$, and hence the same argument could be used to establish the monotonicity of ω_{f_b}, i.e., the conclusion of Conjecture 5.1, whenever $b \ge \frac{1}{2}(5+\nu_1)-1 = 1.360$. It can be checked numerically, however, that $\inf_{t\in\mathbb{R}} p(\varepsilon, t) < 0$ whenever $2 < \varepsilon \le 4.660$. Lemma 4.12 therefore is incapable of establishing Conjecture 5.1 for $0 < b \le 1.330$.

(ii) Numerical evidence strongly suggests that $p(\varepsilon, t) \ge at^4 e^{2\varepsilon t}$ for an appropriate $a > 0$ and all $\varepsilon \ge 5$, $t \in \mathbb{R}$. Obviously, such a lower bound on p, if indeed correct, implies (5.13). In the light of this, an alternative proof of Lemma 5.3 might be provided using rigorous (or validated) numerics; see, e.g., [2, 33] for context. For $\varepsilon \ge 5$, it is easy to see that $p(\varepsilon, t)/t^4 > 0$ whenever $\varepsilon \ge 34$ or $|t| \ge 2$. Thus in order to prove Lemma 5.3, one only has to rigorously verify that $\min_{\mathbb{A}} p/t^4 > 0$ for the compact rectangle $\mathbb{A} = [5, 34] \times [-2, 2]$; see Fig. 8. The reader may want to notice that usage of rigorous numerics or other forms of computer assistance is not uncommon for problems of a similar flavour in non-linear analysis; see, e.g., [1, 6].

(iii) Given any specific *rational* $b \ge \frac{3}{2}$, or $\varepsilon \ge 5$, the conclusion of Lemma 5.3 may be arrived at in yet another way: Letting $\varepsilon = m/n \ge 5$ with coprime $m, n \in \mathbb{N}$, notice that

$$\frac{n^3 p(m/n, nt)}{(e^t - 1)^4} = p_{m,n}(e^t) \quad \forall t \in \mathbb{R},$$

with the appropriate polynomial $p_{m,n}$ with integer coefficients and degree $4m - n - 4$. Using classical algebra tools, it may be straightforward to see directly that $p_{m,n}(s) > 0$ for all $s \in \mathbb{R}^+$. For example, take $b = \frac{3}{2}$, hence $\varepsilon = 5$, so $m = 5$, $n = 1$, and a short calculation yields

$$\begin{aligned} p_{5,1}(s) = {} & 30s^{15} + 120s^{14} + 300s^{13} + 600s^{12} + 798s^{11} + 807s^{10} + 540s^9 - 15s^8 \\ & - 870s^7 - 1281s^6 - 1164s^5 - 435s^4 + 540s^3 + 1395s^2 + 1008s + 252. \end{aligned} \tag{5.18}$$

By Descartes' rule, $p_{5,1} = 0$ has precisely zero or two real roots (counted with multiplicities) on $\mathbb{R}^+$, and the rough estimate implied by (5.18),

$$p_{5,1}(s) \ge 15\left(213s^9 \min\{1, s\}^6 - 251s^4 \max\{1, s\}^4 + 213 \min\{1, s\}^3\right) \quad \forall s \in \mathbb{R}^+,$$

can be used to show that in fact $p_{5,1}(s) > 0$ for all $s \in \mathbb{R}^+$. Thus $p(5, t) \ge 0$ for all $t \in \mathbb{R}$, and hence $\omega_{f_{3/2}}$ is decreasing on $]0, 1[$.

As a consequence of Lemma 5.3, Theorem 4.11 together with (5.1) and (5.2) yields

$$\#\left(\left(\mathbb{N} \setminus \{1\}\right) \cap \left]2\frac{b+1}{b+2}, \sqrt{b+1}\right[\right) = \left\lceil \sqrt{b+1} \right\rceil - 2$$

as the number of different non-circular Jordan solutions of $\kappa = r^b$ modulo rotations, whenever $b \ge \frac{3}{2}$; here $\lceil a \rceil$ denotes the smallest integer not smaller than $a \in \mathbb{R}$. By means of an obvious rescaling, the results of this section so far can be summarized and slightly extended as follows.

Theorem 5.6 *Let $a \in \mathbb{R}^+$, $b \in \mathbb{R}$, and assume that $\mathcal{C}$ is a Jordan solution of $\kappa = ar^b$. Then $\mathcal{C}$ is oriented counter-clockwise, and the following hold:*

(i) if $b \le 3$ and $b \ne -1, 0$ then $[\mathcal{C}]$ is the circle with radius $a^{-1/(b+1)}$ centered at 0;
(ii) if $b = -1$ then $a = 1$, and $[\mathcal{C}]$ is a circle centered at 0;

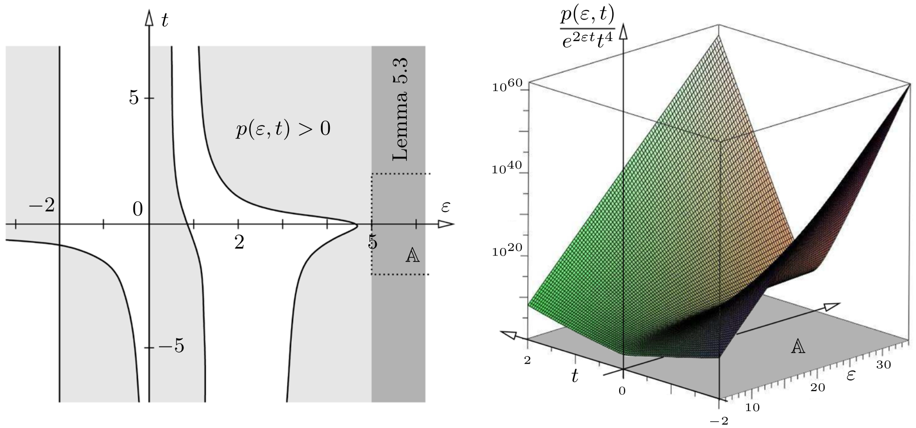

Fig. 8 Solid black curves indicate the zero locus of $p = p(\varepsilon, t)$ given by (5.12), with the set $\{p > 0\}$ shown in grey (left). Plotting the real-analytic function $p(\varepsilon, t)e^{-2\varepsilon t}/t^4$ suggests that $p(\varepsilon, t) \geq at^4e^{2\varepsilon t}$ for all $\varepsilon \geq 5$, $t \in \mathbb{R}^+$, where a may be as large as $10^{3.416} = 2606$ (right)

(iii) if $b = 0$ *then* $[\mathcal{C}]$ *is a circle with radius* a^{-1}*;*
(iv) if $b > 3$ *then either* $[\mathcal{C}]$ *is the circle with radius* $a^{-1/(b+1)}$ *centered at* 0*, or else* $[\mathcal{C}]$ *is non-circular,* $[\mathcal{C}] = a^{-1/(b+1)}e^{i\vartheta}[\mathcal{C}_{f_b,s}]$ *for some* $\vartheta \in \mathbb{R}$ *and a unique* $s \in \mathbb{O}_{f_b}$*, with* $\mathbb{O}_{f_b} \subset\ ]0, 1[$ *containing precisely* $\lceil\sqrt{b+1}\,\rceil - 2$ *elements.*

Proof of Lemma 4.12 Replacing $c \in \mathcal{C}$ by $a^{1/(b+1)}c$, it can be assumed that $a = 1$, provided that $b \neq -1$. Thus only the assertion regarding $b = -1$ requires further justification. For $f(s) = a/s$ with $a \in \mathbb{R}^+$, it is readily seen that $\operatorname{Per}\Phi_f = \varnothing$ if $a > 1$. By contrast, if $a < 1$ then 0 is a center, by Proposition 2.2, and every other orbit is periodic and twisted. In either case, $\mathcal{C}$ is not Jordan, by Theorem 3.13, so necessarily $a = 1$. $\square$

Notice that Theorem 5.6 asserts in particular that $\kappa = ar^b$ with $a \in \mathbb{R}^+$ has no non-circular Jordan solution when $b \leq 3$, but has, modulo rotations, precisely $n \in \mathbb{N}$ different such solutions when $n^2 + 2n < b \leq n^2 + 4n + 3$. For instance, $\kappa = r^4$ and $\kappa = r^9$ have precisely one and two non-circular Jordan solutions, respectively, modulo rotations; see Figs. 1 and 9.

Remark 5.7 As mentioned already in the Introduction, it is a well-documented empirical observation that the oval shapes of worn stones never seem to be exact ellipses, but rather appear to be a bit bulkier. In this regard, the reader may find it interesting to note that none of the non-circular ovals $[\mathcal{C}_{f_b,s}]$ of Theorem 5.6(iv) is an ellipse either. Indeed, suppose that for a Jordan solution $\mathcal{C}$ of $\kappa = ar^b$ with $a \in \mathbb{R}^+$ the set $[\mathcal{C}]$ was an ellipse with semi-axes $A \geq B > 0$. If $b \geq 0$ then $A/B^2 = aA^b$ as well as $B/A^2 = aB^b$, thus $a^2A^4 = B^{2(1-b)} = a^{b-1}A^{(1-b)^2}$, and hence $a^{3-b} = A^{(b-3)(b+1)}$. It follows that either $a = A^{-b-1}$ and consequently $A = B$, or else $b = 3$. By Theorem 5.6, $A = B$ in any case, i.e., $[\mathcal{C}]$ is a circle. (Usage of Theorem 5.6, though convenient, is not essential here: A simple calculation shows directly that any ellipse solving $\kappa = ar^3$ necessarily is a circle.) A similar argument applies when $b < 0$.

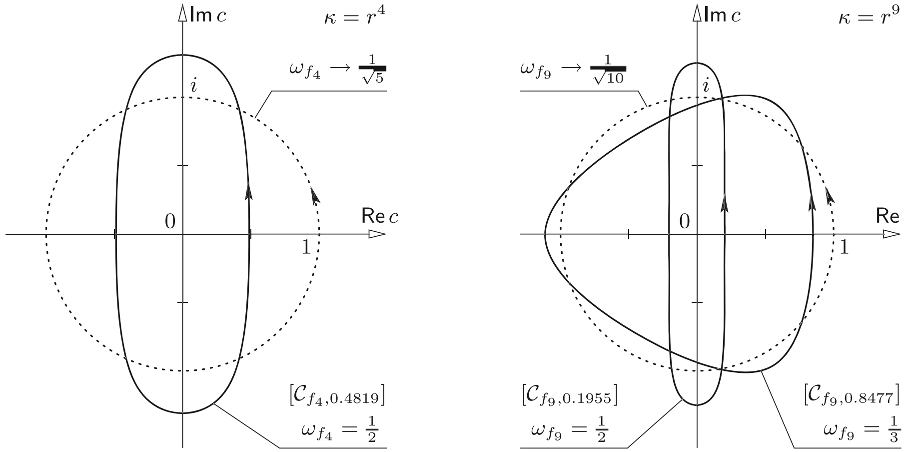

Fig. 9 Apart from the unit circle (dotted), and modulo rotations, $\kappa = r^b$ has precisely one (non-circular) Jordan solution when $b = 4$ (left; see also Fig. 1), and has precisely two such solutions when $b = 9$ (right)

Supplement: The limiting shapes in [5] revisited

As noted earlier, the results of this section are quite specific to the monomial family. However, the basic tools developed earlier in order to obtain these results, notably the auxiliary planar flow Φ_f (Sect. 2), geometric correspondances (Sect. 3), and analytic estimates (Sect. 4), all may be useful in other contexts as well. This supplementary section briefly describes one such context, motivated by the classification of limiting shapes for isotropic curve flows in [5]. As the author intends to give a detailed account elsewhere, only an outline is presented here that highlights the similarity to the main results of the present article; as such, a few non-essential assumptions are made to simplify the exposition.

Let $g : \mathbb{R}^+ \to \mathbb{R}$ be smooth, and assume for convenience that g is increasing and $g(0+) \geq 0$. Fix any $G : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ with $G'(s) = 2s - 2/g(|s|)$ for all $s \in \mathbb{R} \setminus \{0\}$. Note that G is convex on $\mathbb{R}^+$, with a global non-degenerate minimum at s_g, where $s_g > 0$ is the unique solution of $sg(s) = 1$. This usage of the symbol s_g is consistent with earlier usage of s_f. Assume henceforth that $G(s_g) = 0$. On $\mathbb{C} \setminus i\mathbb{R}$ consider an ODE for $z = z(t)$ very similar to (2.1),

$$\dot{z} = izg(|\mathrm{Re}\, z|) - i. \tag{5.19}$$

Note that $G(\mathrm{Re}\, z) + (\mathrm{Im}\, z)^2$ is a first integral of (5.19). While (5.19) does not in general generate a (global, topological) flow on $\mathbb{C}$, unlike (2.1), it does generate a flow, henceforth denoted Ψ_g, on the open convex set $\mathbb{A}_g := \{z \in \mathbb{C} : \mathrm{Re}\, z > 0, G(\mathrm{Re}\, z) + (\mathrm{Im}\, z)^2 < G(0+)\}$. For instance, if g can be extended smoothly (or merely as a Lipschitz function) to $s = 0$ then $\mathbb{A}_g = \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$. In any case, every $z \in \mathbb{A}_g$ is a periodic point of Ψ_g, and $\mathrm{Fix}\, \Psi_g = \{s_g\}$. Similarly to (2.5), define

$$\nu_g(z) = \frac{1}{2\pi} \int_0^{T_g(z)} g(|\mathrm{Re}\, \Psi_g(t, z)|)\, \mathrm{d}t \quad \forall z \in \mathbb{A}_g.$$

Thus $\nu_g(s_g) = 0$ whereas $\nu_g(z) > 0$ for every $z \in \mathbb{A}_g \setminus \{s_g\}$. In analogy to (2.7) it is readily confirmed that

$$\lim_{z \to s_g} \nu_g(z) = \sqrt{\frac{2}{G''(s_g)}} = \frac{1}{\sqrt{1 + s_g^2 g'(s_g)}} \in]0, 1[.$$

Next, similarly to Sect. 3 say that an oriented smooth curve $\mathcal{C}$ is a solution of

$$\kappa = g\big(\big|\mathsf{Re}\,(z\overline{n})\big|\big), \tag{5.20}$$

if $\kappa_c(t) = g\big(\big|\mathsf{Re}\,\big(c(t)(-i)\overline{\dot{c}(t)}\big)\big|\big) = g\big(\big|\mathsf{Im}\,\big(\overline{c(t)}\dot{c}(t)\big)\big|\big)$ for all $t \in \mathbb{J}_c$ with $\overline{c(t)}\dot{c}(t) \notin \mathbb{R}$, where c is some (and hence any) element of $\mathcal{C}$. By the same calculation as in Sect. 3, if $\mathcal{C}$ is a solution of (5.20) and $c \in \mathcal{C}$ then z_c given by (3.2) solves (5.19). Thus, just as for (3.1), a correspondance can be established between maximal solutions of (5.20) modulo O(2)-congruence on the one hand and orbits of Ψ_g on the other hand. (A careful analysis needs to pay attention to the possibility of $\mathsf{Im}\,(\overline{c}\dot{c}) = 0$, a situation reflected for $g(0+) = 0$ by the invariance in (5.19) of the imaginary axis.) In particular, it makes sense to define $\nu_g(\mathcal{C}) = \nu_g\big(z_c(0)\big)$ for any $c \in \mathcal{C}$. With this, if $\mathcal{C}$ is a Jordan solution of (5.20) other than the (counter-clockwise oriented) circle with radius s_g centred at 0 then $\nu_g(\mathcal{C}) = 1/n$ for some $n \in \mathbb{N}$. As in the case of (3.1), it is natural to find all Jordan solutions of (5.20), and to do so by determining the range of ν_g as accurately as possible. With $s^* > s_g$ defined uniquely by $G(s^*) = G(s)$ for every $0 < s < s_g$, as in Sect. 4,

$$\nu_g(s) = \frac{1}{\pi} \int_s^{s^*} \frac{\mathrm{d}u}{\sqrt{G(s) - G(u)}} \quad \forall 0 < s < s_g. \tag{5.21}$$

This usage of the symbol s^* is slightly inconsistent with its earlier usage. However, rather than confusing the reader, it will hopefully highlight the parallels between the analyses of (3.1) and (5.20), respectively. Note that (5.21), though very similar in spirit to (4.1), is considerably simpler, in at least two respects: On the one hand, utilizing the smooth positive function $K := G/(G')^2$, together with (a trivially adjusted version of) Proposition 4.14, yields

$$\begin{aligned} \nu_g'(s) &= \frac{G'(s)}{2\pi G(s)} \int_s^{s^*} \frac{G'(u)K'(u)}{\sqrt{G(s) - G(u)}}\,\mathrm{d}u \\ &= \frac{G'(s)}{\pi G(s)} \int_s^{s^*} \sqrt{G(s) - G(u)}\,K''(u)\,\mathrm{d}u \quad \forall 0 < s < s_g, \end{aligned} \tag{5.22}$$

which is far less unwieldy than (4.8). For instance, (5.22) makes obvious the known fact that $s \mapsto \nu_g(s)$ is monotone whenever K is convex or concave [9, 11, 13, 29, 37]. Applying Proposition 4.14 once more yields, for every $0 < s < s_g$,

$$\begin{aligned} \nu_g''(s) = &\frac{1}{2\pi G(s)K(s)} \int_s^{s^*} \frac{G(u)K''(u)}{\sqrt{G(s) - G(u)}}\,\mathrm{d}u \\ &- \frac{G'(s)K'(s)}{2\pi G(s)K(s)} \int_s^{s^*} \sqrt{G(s) - G(u)}\,K''(u)\,\mathrm{d}u. \end{aligned}$$

In analogy to Proposition 4.7, therefore, ν_g is smooth on $]0, s_g[$, with

$$\nu_g(s_g-) = \sqrt{\frac{2}{G''(s_g)}}, \quad \nu_g'(s_g-) = 0,$$
$$\nu_g''(s_g-) = \frac{5G^{(3)}(s_g)^2 - 3G''(s_g)G^{(4)}(s_g)}{12\sqrt{2}G''(s_g)^{5/2}}. \tag{5.23}$$

On the other hand, notice that $\pi\nu_g(s)$ can be interpreted as the *true* minimal period of the point $s \in \mathbb{A}_g$ in the planar Hamiltonian flow on $\mathbb{A}_g$ generated by

$$\dot{w} = 2\,\mathrm{Im}\, w - iG'(\mathrm{Re}\, w), \tag{5.24}$$

which has s_g as a non-degenerate center; cf. [5,Sec. 2]. This interpretation makes ν_g directly amenable to the substantial literature on 1-DOF Hamiltonian systems, notably on the periods of such systems; see, e.g., [9, 11–15, 31, 35, 36, 38]. By contrast, much of the delicate analysis from Proposition 4.7 onward has been necessitated by the fact that no similar interpretation seems to exist for ω_f; see also Remark 2.8.

Finally, to illustrate the above for a familiar example, consider once again the monomial family, that is, let $g = f_b$ with $b \in \mathbb{R}^+$. Here, for all $s \in \mathbb{R}^+$,

$$G(s) = \begin{cases} s^2 + \dfrac{2}{b-1}s^{1-b} - \dfrac{b+1}{b-1} & \text{if } b \neq 1, \\ s^2 - 2\log s - 1 & \text{if } b = 1, \end{cases}$$

and $\mathbb{A}_{f_b} = \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$ precisely if $b \geq 1$. Also, $s_{f_b} = 1$, and (5.23) yields

$$\nu_{f_b}(1-) = \frac{1}{\sqrt{b+1}}, \quad \nu_{f_b}'(1-) = 0, \quad \nu_{f_b}''(1-) = \frac{b(b-3)}{12\sqrt{b+1}}.$$

Thus, as $s \to 1$ the function ν_{f_b} attains a non-degenerate maximum (respectively, minimum) if $0 < b < 3$ (respectively, if $b > 3$). In addition, a straightforward calculation shows that

$$\nu_{f_b}(0+) = \frac{1}{1 + \min\{b, 1\}} \quad \forall b \in \mathbb{R}^+.$$

As a consequence, $(b-3)\big(\nu_{f_b}(0+) - \nu_{f_b}(1-)\big) > 0$ for every $b \in \mathbb{R}^+ \setminus \{3\}$, which certainly makes it plausible to speculate that ν_{f_b} is increasing (respectively, decreasing) on $]0, 1[$ if $0 < b < 3$ (respectively, if $b > 3$). Notice how this is the precise analogue of Conjecture 5.1. Unlike for the latter, it is not hard to prove this speculation correct in its entirety. Given that (5.21) is considerably simpler than (4.1), as noted above, this may not come as a complete surprise [5].

Proposition 5.8 *On* $]0, 1[$, *the function* ν_{f_b} *is increasing for* $0 < b < 3$, *and decreasing for* $b > 3$.

As Proposition 5.8 suggests, the case $b = 3$ is somewhat special: Indeed, here $s^* = 1/s$ for every $0 < s < 1$, and hence

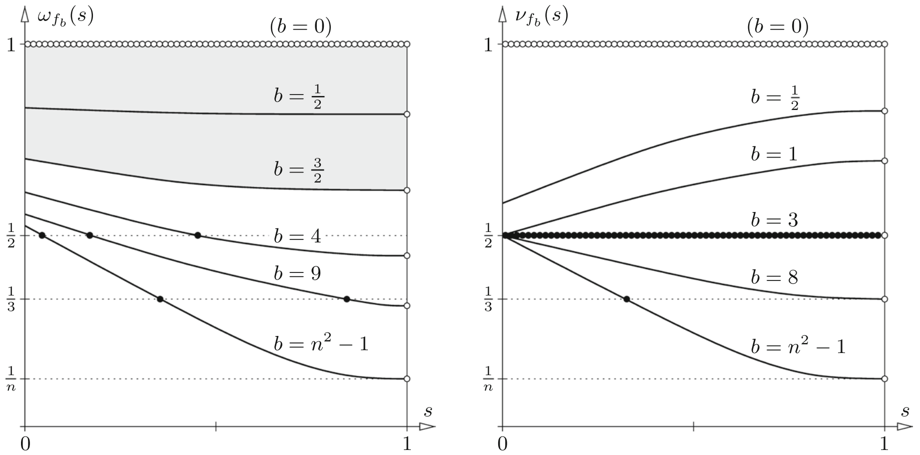

Fig. 10 Finding all Jordan solutions of $\kappa = r^b$ (left) and $\kappa = |\mathrm{Re}\,(z\overline{n})|^b$ (right) with $b \in \mathbb{R}^+$, by qualitatively graphing ω_{f_b} and ν_{f_b}, respectively. At the time of this writing, monotonicity (in s) of ω_{f_b} for $0 < b < \frac{3}{2}$ is conjectural only (grey region). Solid black dots indicate non-circular Jordan solutions, while circles represent circular solutions with radius 1

$$\begin{aligned}\nu_{f_3}(s) &= \frac{1}{\pi}\int_s^{1/s} \frac{\mathrm{d}u}{\sqrt{(s-1/s)^2-(u-1/u)^2}} \\ &= \frac{1}{2\pi}\int_{s^2}^{1/s^2} \frac{\mathrm{d}u}{\sqrt{(u-s^2)(1/s^2-u)}} = \frac{1}{2} \quad \forall 0 < s < 1.\end{aligned}$$

In other words, the center 1 of (5.24) is *isochronous* for $g = f_3$. This "surprising affine invariance property" [5] reflects the fact that the phase portrait of (5.24) for $g = f_3$ is invariant under the diffeomorphism $z \mapsto (\mathrm{Re}\,z)^{-1} - i\,\mathrm{Im}\,z$ of $\mathbb{A}_{f_3}$. Further analysis shows that every orbit $\Psi_{f_3}(\mathbb{R}, s)$ with $s \in \mathbb{R}^+$ corresponds to a (counter-clockwise oriented) ellipse with semi-axes s, $1/s$. Thus, every maximal solution of $\kappa = |\mathrm{Re}\,(z\overline{n})|^3$ is an ellipse centered at 0 with interior area π.

As a consequence of Proposition 5.8, for every $0 < b \le 8$ with $b \neq 3$ the only Jordan solution of

$$\kappa = |\mathrm{Re}\,(z\overline{n})|^b \tag{5.25}$$

is the (counter-clockwise oriented) unit circle, whereas for $b > 8$ there exist precisely $\lceil\sqrt{b+1}\rceil - 3$ different non-circular Jordan solutions of (5.25), modulo rotations; see Fig. 10 and [5,Thm. 5.1].

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