More Grade School Triangles

Arno Berger

Abstract. Planar triangles with each side having a positive integer length are among the simplest geometrical objects imaginable. What can be said about the angles of such triangles? In particular, are these angles rational or at least algebraic numbers when measured in degrees? This article demonstrates that the answer in general is negative, except for three distinguished families of triangles. One family is well known since antiquity: With the largest angle equal to 90 degrees, it simply is the family of Pythagorean right triangles. Though not nearly as well-known, the other two families also deserve to be part of every geometry teacher's toolkit.

1. INTRODUCTION. Planar triangles are a bedrock of the geometry curriculum and have been for centuries, if not millennia. Besides their natural beauty, one main reason for this longevity is that they provide an ideal playground for anyone learning to translate *pictures* into *numbers*. Aimed at making this important transition from geometry to trigonometry, algebra, and number theory easy and enjoyable, the triangles encountered in the classroom often yield numbers that are especially simple. For instance, all three side lengths may be positive integers, or all three angles, measured in degrees, may be rational or perhaps even both. Correspondingly, right triangles with integer side lengths as well as isosceles triangles with rational angles figure prominently in grade school geometry. In fact, such is their prominence that one is led to suspect that no other simple triangles exist at all. As the present article is going to demonstrate, this suspicion is largely correct. The article is inspired by the charming paper [**3**] that focuses on simple *right* triangles.

To fix notation and terminology, denote by ℓ_1, ℓ_2, and ℓ_3 the shortest, middle, and longest side length, respectively, of a nondegenerate planar triangle; hence, $0 < \ell_1 \leq \ell_2 \leq \ell_3 < \ell_1 + \ell_2$. Arguably, the simplest situation arises when each ℓ_j is a positive integer multiple of a common unit length. For lack of a widely accepted term in this regard, call any such triangle *elementary*. In other words, with the appropriate positive integers n_j, an elementary triangle satisfies $\ell_1/\ell_2 = n_1/n_2$ and $\ell_2/\ell_3 = n_2/n_3$, henceforth written simply as $\ell_1 : \ell_2 : \ell_3 = n_1 : n_2 : n_3$.

Ideally, when measured in degrees, the three angles of a student's favorite triangle are simple numbers as well; at the very least, they should be rational. Denote by δ_j the angle vis-à-vis ℓ_j, hence $0 < \delta_1 \leq \delta_2 \leq \delta_3$ and $\delta_1 + \delta_2 + \delta_3 = 180$, and call the triangle $\Delta = \langle \delta_1, \delta_2, \delta_3 \rangle$ *rational* if each δ_j is a rational number. (Notice that $\langle \delta_1, \delta_2, \delta_3 \rangle$ and $\ell_1 : \ell_2 : \ell_3$, while uniquely determining one another, determine a triangle only up to similarity. Strictly speaking, therefore, $\Delta = \langle \delta_1, \delta_2, \delta_3 \rangle$ is the *similarity type* of a planar triangle; to avoid clumsy terminology, however, simply refer to Δ as a *triangle*.)

Naturally, it may be every geometry student's (and teacher's) dream to have a rich supply of triangles that are both elementary and rational. This, alas, is too much to hope for: The only rational elementary triangle is equilateral; see, e.g., [**2**, Cor. 6] or [**5**, p. 228]. Thus, in order for there to be any nonequilateral triangle at all with simple side lengths *and* angles, either the rationality or the elementariness assumption has to be relaxed. On the one hand, every Pythagorean right triangle, for instance, is elementary with $\delta_3 = 90$, but it turns out that neither δ_1 nor δ_2 can be rational. On the other hand,

http://dx.doi.org/10.4169/amer.math.monthly.124.4.324
MSC: Primary 11A99, Secondary 97G40

rational triangles with $\delta_3 = 90$ do exist—provided that the lengths ℓ_j are allowed to be quadratic irrational numbers, in which case of course the triangle may not quite be elementary. In fact, the main result of [**3**] asserts that there are exactly three triangles with this property (see Figure 4 below), and they are often encountered in grade school geometry.

The present article complements and extends [**3**] by venturing beyond right triangles. One main result (Theorem 7 below) shows that every elementary triangle with at least one rational angle belongs to one of only three distinguished families of triangles. To catch a first glimpse of these families, consider Δ_1, Δ_2, and Δ_3 with $\ell_1 : \ell_2 : \ell_3$ given by

$$3:4:5\,, \quad 3:5:7\,, \quad \text{and} \quad 3:7:8\,, \tag{1}$$

respectively. Do the three elementary triangles in (1) have anything interesting in common? Being elementary alone is clearly not that remarkable a property: Given any positive integers $n_1 \le n_2 \le n_3 < n_1 + n_2$, taking $\ell_1 : \ell_2 : \ell_3 = n_1 : n_2 : n_3$ yields an elementary triangle, and all elementary triangles arise that way. What is interesting about Δ_1, though, is that $3^2 + 4^2 = 5^2$, and hence, Δ_1 is a *right* (or *Pythagorean*) triangle, so $\delta_3 = 90$. In the case of Δ_2, it follows from $3^2 + 5^2 + 3 \cdot 5 = 7^2$ and the law of cosines that $\delta_3 = 120$. Similarly, $3^2 + 8^2 - 3 \cdot 8 = 7^2$, and so $\delta_2 = 60$ for Δ_3. Triangles with $\delta_2 = 60$ or $\delta_3 = 120$ are referred to informally as *pseudo-Pythagorean*; see Section 3 for precise definitions. In summary, each triangle in (1) is elementary and has one *rational* (in fact, integer) angle. (When measured in radians, one angle is a *rational multiple* of π; for simplicity and in adherence to the tradition of grade school geometry, all angles in this article are measured in degrees.)

Now, the remarkable feature of (1) is that, in a sense, these three triangles represent all possible ways a rational angle may ever occur in an elementary triangle.

Claim 1. Let $\Delta = \langle \delta_1, \delta_2, \delta_3 \rangle$ be an elementary triangle. If one angle δ_j is rational, then either $\delta_2 = 60$ or $\delta_3 \in \{90, 120\}$, i.e., Δ is Pythagorean or pseudo-Pythagorean.

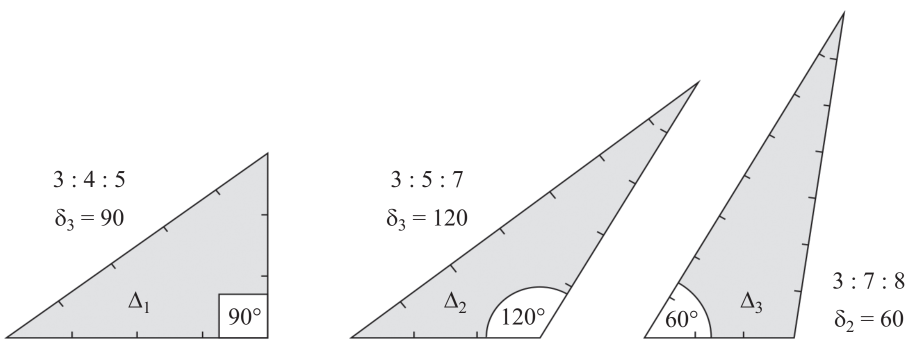

Figure 1. Realizations with equal area of the triangles in (1). Claim 1 asserts that every elementary triangle with at least one rational angle is either Pythagorean (left) or pseudo-Pythagorean.

Proving a stronger, more precise version of Claim 1 is one main goal of this article; see Section 3. In preparation for this, Section 2 reviews simple algebraic properties of trigonometric numbers. With their special role established by Theorem 7, Pythagorean and pseudo-Pythagorean triangles are given a unified treatment in Section 4; the ensuing complete classification of such triangles (Theorem 8) is another main result of this article. The concluding Section 5 briefly explores how the elementariness assumption (rather than the rationality assumption, as in Claim 1) might be relaxed. For instance, call a triangle *almost elementary* if at least two of the numbers u_j in $\ell_1 : \ell_2 : \ell_3 = u_1 : u_2 : u_3$ are integers. Every isosceles triangle, for example, is almost elementary since either $u_1 = u_2$ or $u_2 = u_3$ can be scaled so as to be an integer in this case. (Equivalently, an appropriate unit length can be chosen.) This example yields an infinite supply of rational almost elementary triangles. Perhaps surprisingly, as shown in Section 5, there exists only a single rational almost elementary triangle that is not isosceles.

Claim 2. Let $\Delta = \langle \delta_1, \delta_2, \delta_3 \rangle$ be an almost elementary triangle. If Δ is rational, then either $\delta_1 = \delta_2$ or $\delta_2 = \delta_3$, i.e., Δ is isosceles, or else $\Delta = \langle 30, 60, 90 \rangle$.

2. TRIGONOMETRIC NUMBERS.

As a first step toward establishing Claim 1, consider any triangle $\Delta = \langle \delta_1, \delta_2, \delta_3 \rangle$, with corresponding side lengths ℓ_1, ℓ_2, and ℓ_3. By the law of cosines,

$$2\cos\delta_1^\circ = \frac{\ell_3}{\ell_2} + \frac{\ell_2}{\ell_3} - \frac{\ell_1}{\ell_2} \cdot \frac{\ell_1}{\ell_3}, \tag{2}$$

and similarly for $\cos\delta_2^\circ$ and $\cos\delta_3^\circ$. Thus, if Δ is elementary, then each number $\cos\delta_j^\circ$ is rational. Here and throughout, usage of the degree symbol ($^\circ$) is reminding the reader that arguments of trigonometric functions are interpreted as angles and measured in degrees; formally, $\cos\delta^\circ = \cos(\pi\delta/180)$. In view of Claim 1, the crucial question is whether in an elementary triangle any angle δ_j may be rational as well. In other words, can the two numbers δ_j and $\cos\delta_j^\circ$ both be rational? Similarly, Claim 2 turns out to hinge on whether the three numbers δ_j, δ_k, and $\cos\delta_j^\circ/\cos\delta_k^\circ$ can all be rational for $j \neq k$; see Section 5. Fortunately, these questions can be resolved easily (Corollaries 4 and 5 below), but in order to do so, first a bit of algebraic notation and terminology needs to be reviewed.

Denote the sets of all positive integers, integers, rational, and complex numbers by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$, respectively, and the empty set by $\varnothing$, as usual. For any number $w \in \mathbb{C}$ and any set $\mathcal{W} \subset \mathbb{C}$, let $w\mathcal{W} = \{wz : z \in \mathcal{W}\}$. Recall that $w \in \mathbb{C}$ is *algebraic* if there exists a nonconstant polynomial P with integer coefficients such that $P(w) = 0$; if w is not algebraic, then it is *transcendental*. Thus, for instance, $\sqrt{5} + 1$ and $\sqrt{5 - \sqrt{5}}$ both are algebraic, as they solve $w^2 - 2w - 4 = 0$ and $w^4 - 10w^2 + 20 = 0$, respectively. Well-known examples of transcendental numbers include e, π, and $2^{\sqrt{2}}$; see also Proposition 6 below. Note that if a number is algebraic, then so is its complex conjugate. For convenience, denote by $\mathbb{A} \subset \mathbb{C}$ the set of all algebraic numbers. With a bit of work, it can be shown that $\mathbb{A}$ in fact is a field; see, e.g., [**12**, Thm. 7.2]. Hence, if w is algebraic, then so is w^n for every $n \in \mathbb{N}$, but also the real and imaginary parts of w. Specifically, notice that $\cos r^\circ$ with $r \in \mathbb{Q}$ is algebraic since it is the real part of the algebraic number $e^{\pi\imath r/180}$. Algebraic properties of trigonometric numbers such as $\cos r^\circ$, $\sin r^\circ$, or $\tan r^\circ$ have long been of interest, not least to authors and readers of the MONTHLY [**2, 3, 4, 8, 10, 13, 18, 19, 20**]. For simplicity, in what follows only numbers $\cos r^\circ$ are considered, though analogous considerations pertain to $\sin r^\circ$ and $\tan r^\circ$ as well. Finally, recall that the field $\mathbb{C}$ is a linear space over any of its subfields, such as, e.g., $\mathbb{Q}$ or $\mathbb{A}$. In particular, therefore, it makes sense to ask whether $\mathcal{W} \subset \mathbb{C}$ is *linearly (in)dependent over* $\mathbb{Q}$, or $\mathbb{Q}$-*(in)dependent*, for short. A simple linear algebra exercise shows that for $w \in \mathbb{C} \setminus \{0\}$ and $z \in \mathbb{C}$ the quotient z/w is rational if and only if $\{w, z\}$ is $\mathbb{Q}$-dependent.

Using the above terminology, the two questions encountered earlier can be concisely rephrased as follows: For which rational r_1 is $\{1, \cos r_1^\circ\}$ $\mathbb{Q}$-dependent? And, for which rational r_1, r_2 is $\{\cos r_1^\circ, \cos r_2^\circ\}$ $\mathbb{Q}$-dependent? Note that if either set is $\mathbb{Q}$-dependent, then so is $\{1, \cos r_1^\circ, \cos r_2^\circ\}$, and hence, a complete analysis of the latter allows one to answer both questions at once. For which $r_1, r_2 \in \mathbb{Q}$, then, is the set $\{1, \cos r_1^\circ, \cos r_2^\circ\}$ $\mathbb{Q}$-dependent? As every student of trigonometry discovers very early on, if r_1 is an integer multiple of 60 or 90, then $\cos r_1^\circ$ actually is rational, and hence, $\{1, \cos r_1^\circ\}$ is $\mathbb{Q}$-dependent. Also, if $\{r_1 - r_2, r_1 + r_2\} \cap 180\mathbb{Z} \neq \varnothing$, i.e., if $r_1 - r_2$ or $r_1 + r_2$ is an integer multiple of 180, then $|\cos r_1^\circ| = |\cos r_2^\circ|$, and hence, $\{\cos r_1^\circ, \cos r_2^\circ\}$ is $\mathbb{Q}$-dependent. Moreover,

$$4\cos 36^\circ - 4\cos 72^\circ = \sqrt{5} + 1 - \left(\sqrt{5} - 1\right) = 2\,.$$

As it turns out, the only ways for $\{1, \cos r_1^\circ, \cos r_2^\circ\}$ to be $\mathbb{Q}$-dependent are the ones just described. While this fact is stated here without proof, the interested reader is referred to [**1**] for a simple ad hoc derivation involving only the divisibility of polynomials with integer coefficients and to [**6, 9, 11**] for a systematic study of vanishing sums of roots of unity utilizing algebraic number theory.

Theorem 3. *Let $r_1, r_2 \in \mathbb{Q}$, and assume that $\{r_1 - r_2, r_1 + r_2\} \cap 180\mathbb{Z} = \varnothing$. Then the following are equivalent:*

(i) *The set $\{1, \cos r_1^\circ, \cos r_2^\circ\}$ is $\mathbb{Q}$-dependent;*

(ii) *Both numbers r_1, r_2 are integer multiples of* 36*, or at least one of them is an integer multiple of* 60 *or* 90*.*

This article only utilizes Theorem 3 via two immediate corollaries. Though already recorded in [**18**], the first corollary sometimes is attributed to [**12**] as *Niven's Theorem*.

Corollary 4. *Let $r \in \mathbb{Q}$. Then the following are equivalent:*

(i) $\cos r^\circ \in \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right\}$*;*

(ii) $\cos r^\circ$ *is rational;*

(iii) r *is an integer multiple of* 60 *or* 90*.*

Proof. Plainly, (i)⇒(ii). To see (ii)⇒(iii), assume that $\cos r^\circ$ is rational, and let $r_1 = r$. Suppose r_1 was not an integer multiple of 60 or 90. Then it would be possible to choose $r_2 \in \mathbb{Q}$ with $0 < r_2 < 1$ so small that $\{r_1 - r_2, r_1 + r_2\} \cap 180\mathbb{Z} = \varnothing$. By Theorem 3, the set $\{1, \cos r_1^\circ, \cos r_2^\circ\}$ would be $\mathbb{Q}$-independent, which clearly contradicts the rationality of $\cos r_1^\circ$. Hence, r_1 must be an integer multiple of 60 or 90. To establish (iii)⇒(i), simply note that the function $r \mapsto \cos r^\circ$ is even, has period 360, and at $r \in \{0, 60, 90, 120, 180\}$ attains the values listed in (i). ■

Corollary 5. *Let $r_1, r_2 \in \mathbb{Q}$, and assume that $0 < r_1, r_2 < 90$. Then $\cos r_1^\circ / \cos r_2^\circ$ is rational if and only if $r_1 = r_2$.*

Proof. Let $w = \cos r_1^\circ / \cos r_2^\circ$ be rational and suppose $r_1 \neq r_2$. Then $\{r_1 - r_2, r_1 + r_2\} \cap 180\mathbb{Z} = \varnothing$, so by Theorem 3 either both numbers r_1, r_2 are integer multiples of 36 or else one of them is 60. In the first case, w or w^{-1} equals $\left(\sqrt{5}+1\right)/\left(\sqrt{5}-1\right) = \frac{1}{2}\left(\sqrt{5}+3\right)$ and hence is irrational, contradicting the rationality of w. In the second case, assume without loss of generality that $r_1 = 60$, hence $\cos r_1^\circ = \frac{1}{2}$. Then $\cos r_2^\circ$ is rational as well, and so $r_2 = 60 = r_1$, by Corollary 4, contradicting $r_1 \neq r_2$. In summary, if w is rational, then $r_1 = r_2$. The converse is obvious. ■

To conclude this section, recall that for any $r \in \mathbb{Q}$ the trigonometric number $\cos r^\circ$, though rarely rational, always is algebraic. In general, however, as the reader may know or suspect, deciding whether a given complex number is algebraic or transcendental can be very difficult. The following contains two cherished classical results in this regard, to be utilized in the next section.

Proposition 6. *Let $w \in \mathbb{C} \setminus \{0\}$. Then:*

(i) *At least one of the numbers w and e^w is transcendental;*

(ii) *For each $z \in \mathbb{C} \setminus \mathbb{Q}$, at least one of the numbers z, e^w, and e^{zw} is transcendental.*

Note that (i), often referred to as the Hermite–Lindemann theorem, implies the transcendence of e (take $w = 1$) and π (take $w = \pi\imath$), whereas (ii), also known as the Gelfond–Schneider theorem, shows that $2^{\sqrt{2}}$ is transcendental (take $z = \sqrt{2}$ and $w = \ln 2$); see, e.g., [**12**, Ch. 9-10] for details and proofs.

3. ELEMENTARY TRIANGLES. The scene is now set for proving Claim 1 in a somewhat stronger, more precise form. In the proof, the integer solutions of

$$x^2 + \nu xy + y^2 = z^2\,, \tag{3}$$

with $\nu \in \{-1, 0, 1\}$, play a key role. In analogy to the classical Pythagorean triples that correspond to the case $\nu = 0$, call any integer solution (x, y, z) of (3) a ν-*Pythagorean triple* if x, y, and z all are positive; informally, ν-Pythagorean triples for $|\nu| = 1$ are referred to as *pseudo-Pythagorean*. In addition, say a ν-Pythagorean triple is *primitive* if $\gcd(x, y, z) = 1$ and *ordered* if $x \leq y$. For example, the triples $(3, 4, 5)$, $(3, 5, 7)$, and $(3, 8, 7)$ are primitive, ordered, and, as seen in the Introduction, 0-Pythagorean, 1-Pythagorean, and (-1)-Pythagorean, respectively. The significance of ν-Pythagorean triples is highlighted by the following precise form of Claim 1; notice the interchange of the symbols ℓ_2 and ℓ_3 in alternative (iii).

Theorem 7. *Let $\Delta = \langle \delta_1, \delta_2, \delta_3 \rangle$ be an elementary triangle. Then exactly one of the following alternatives applies:*

(i) *$\delta_3 = 90$ but δ_1, δ_2 are both transcendental, and $\ell_1 : \ell_2 : \ell_3 = x : y : z$ for a uniquely determined ordered primitive 0-Pythagorean triple (x, y, z);*

(ii) *$\delta_3 = 120$ but δ_1, δ_2 both are transcendental, and $\ell_1 : \ell_2 : \ell_3 = x : y : z$ for a uniquely determined ordered primitive 1-Pythagorean triple (x, y, z);*

(iii) *$\delta_2 = 60$ but δ_1, δ_3 both are transcendental unless $\delta_1 = \delta_3 = 60$, and $\ell_1 : \ell_3 : \ell_2 = x : y : z$ for a uniquely determined ordered primitive (-1)-Pythagorean triple (x, y, z);*

(iv) *δ_1, δ_2, δ_3 all are transcendental.*

Proof. Fix any realization of the elementary triangle $\Delta = \langle \delta_1, \delta_2, \delta_3 \rangle$. As seen from (2), each number $\cos \delta_j^\circ$ is rational. If $\delta_j \notin \mathbb{Q}$, then, with $w = \pi\imath/180$, the numbers e^w and $e^{w\delta_j} = \cos \delta_j^\circ + \imath \sin \delta_j^\circ$ both are algebraic, and hence, $\delta_j \notin \mathbb{A}$ by Proposition 6(ii). In other words, each angle of an elementary triangle is either rational or transcendental. Since $\delta_1 \leq 60 \leq \delta_3$, it follows from Corollary 4 that $\delta_j \in \mathbb{Q}$ if and only if $\delta_j \in \{60, 90, 120\}$. This leaves the following four alternatives.

(i) $\delta_3 = 90$. If one of the two angles δ_1, δ_2 were rational, then so would be the other, and in fact $\delta_1 = \delta_2 = 60$. This, however, is impossible, as $\delta_1 + \delta_2 = 90$. Hence, δ_1, δ_2 are both transcendental. Moreover, $\ell_1^2 + \ell_2^2 = \ell_3^2$, and since Δ is elementary, there exists a real number $u > 0$ such that $(u\ell_1, u\ell_2, u\ell_3) = (x, y, z)$ for one and only one ordered primitive 0-Pythagorean triple (x, y, z). Clearly, $\ell_1 : \ell_2 : \ell_3 = x : y : z$ in this case.

(ii) $\delta_3 = 120$. As in (i), neither δ_1 nor δ_2 can be rational since that would entail $\{\delta_1, \delta_2\} \subset \{60, 90\}$ and contradict $\delta_1 + \delta_2 = 60$. Also, $\ell_1^2 + \ell_1\ell_2 + \ell_2^2 = \ell_3^2$, and so $\ell_1 : \ell_2 : \ell_3 = x : y : z$ for a unique ordered primitive 1-Pythagorean triple (x, y, z).

(iii) $\delta_2 = 60$. Here, either $\delta_3 = 60$, in which case $\Delta = \langle 60, 60, 60\rangle$ and $\ell_1 : \ell_2 : \ell_3 = 1 : 1 : 1$, or else δ_1, δ_3 both are transcendental. In either case, $\ell_1^2 - \ell_1\ell_3 + \ell_3^2 = \ell_2^2$, and hence, $\ell_1 : \ell_3 : \ell_2 = x : y : z$ for a unique primitive (-1)-Pythagorean triple (x, y, z); since $\ell_1 \leq \ell_3$ by assumption, this triple is ordered.

(iv) If $\delta_2 \neq 60$ and $\delta_3 \notin \{60, 90, 120\}$, then, as seen above, each angle δ_j is transcendental. ■

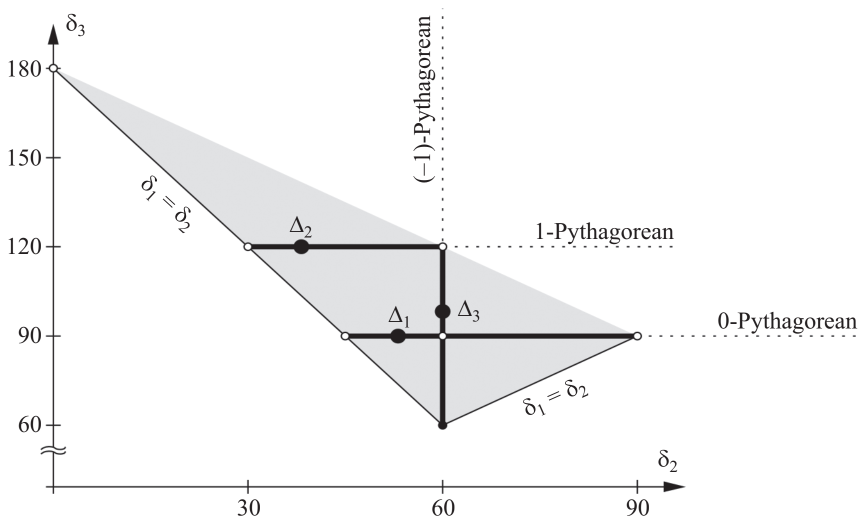

Figure 2. Elementary triangles densely fill the space of all triangles (gray region). However, by Theorem 7 every elementary triangle having at least one algebraic angle lies on one of only three distinguished lines since it is either Pythagorean ($\delta_3 = 90$) or pseudo-Pythagorean ($\delta_2 = 60$ or $\delta_3 = 120$); the triangles of Figure 1 illustrate each case.

In light of the above proof of Theorem 7, it will come as no surprise to the reader that it is alternative (iv) that applies to the overwhelming majority of elementary triangles. In other words, most elementary triangles do not have any algebraic angle at all. Utilizing the classification of ν-Pythagorean triples provided in the next section, an easy counting exercise illustrates this quantitatively. For instance, there exist exactly 94 different elementary triangles for which $\ell_1 : \ell_2 : \ell_3 = n_1 : n_2 : n_3$ with $\max_j n_j \leq 10$. Among these 94 triangles, only one is 0-Pythagorean, one is 1-Pythagorean, and three are (-1)-Pythagorean; see also Figure 3 below. For $\max_j n_j \leq 100$, the respective total is 71,674 elementary triangles, among which a mere 16 are 0-Pythagorean, 14 are 1-Pythagorean, and 27 are (-1)-Pythagorean. Thus, fewer than 0.08 % of all elementary triangles with $\max_j n_j \leq 100$ have even a single algebraic angle. It was this remarkable scarcity of elementary triangles with an algebraic angle that prompted the present article.

Remark. In Theorem 7, if δ_j is measured in radians, then either $\delta_j/\pi \in \{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$ or $\delta_j/\pi \notin \mathbb{A}$. In the latter case, Proposition 6(i) implies that $\delta_j \notin \mathbb{A}$ as well, i.e., the angle δ_j also is transcendental when measured in radians; see [**3**, Fact 2].

4. PYTHAGOREAN AND PSEUDO-PYTHAGOREAN TRIPLES. By Theorem 7, every elementary triangle that has at least one algebraic angle corresponds to an ordered primitive Pythagorean or pseudo-Pythagorean triple. By (2) and Corollary 4, the converse also is true: Every such triple determines an elementary triangle having one angle that is an algebraic number (and in fact equals 60, 90, or 120 degrees). In light of this, every student of geometry will find it useful to have a complete list of ordered primitive ν-Pythagorean triples available. The following theorem provides such a list, thereby extending the Pythagorean case known since antiquity and presented in virtually every basic number theory text; e.g., see [**17**, Sec. 13.1]. The case $\nu = 0$ is included for completeness and also to enable the reader to appreciate the analogy to the cases $\nu = \pm 1$ for which the author has not been able to identify an explicit reference; see, however, [**15, 16**] and the references therein.

Theorem 8. *Let $\nu \in \{-1, 0, 1\}$, and assume that $(x, y, z) \neq (1, 1, 1)$ is an ordered primitive ν-Pythagorean triple. Then there exists a unique pair (m, n) of coprime positive integers with the following properties:*

(i) *For $\nu = 0$, the number $m - n$ is positive and odd, and*

$$x = \min\{2mn, m^2 - n^2\}, \quad y = \max\{2mn, m^2 - n^2\}, \quad z = m^2 + n^2;$$

(ii) *For $\nu = 1$, the number $m - n$ is positive and not divisible by 3, and*

$$x = \min\{2mn + n^2, m^2 - n^2\}, \quad y = \max\{2mn + n^2, m^2 - n^2\},$$
$$z = m^2 + mn + n^2;$$

(iii) *For $\nu = -1$, the number $m - 2n$ is positive and not divisible by 3, and*

$$x = m^2 - 2mn \quad \text{or} \quad x = 2mn - n^2, \quad y = m^2 - n^2,$$
$$z = m^2 - mn + n^2.$$

Conversely, every (x, y, z) given by (i), (ii), and (iii), respectively, with coprime positive integers m, n is an ordered primitive ν-Pythagorean triple.

Proof. Let $(x, y, z) \neq (1, 1, 1)$ be an ordered primitive ν-Pythagorean triple. The challenge here is to demonstrate that (x, y, z) has the form claimed in (i), (ii), and (iii), respectively. Once this is achieved, the converse follows easily.

To begin, note the following simple fact: If $(x, y, z) = r(X, Y, Z)$ for some $r > 0$ and $X, Y, Z \in \mathbb{N}$ with $\gcd(X, Y, Z) = 1$, then $r = 1$. To see this, observe that r is rational, say $r = p/q$ with coprime $p, q \in \mathbb{N}$; hence, the integers qx, qy, and qz all are divisible by p and so are x, y, and z since $\gcd(p, q) = 1$. Similarly, pX, pY, and pZ are divisible by q and so are X, Y, and Z. But then $p = 1$ as $\gcd(x, y, z) = 1$, and $q = 1$ as $\gcd(X, Y, Z) = 1$.

Next, letting $\xi = x/z$ and $\eta = y/z$, rewrite (3) in the form

$$\xi^2 + \nu\xi\eta + \eta^2 = 1. \tag{4}$$

Since ξ, η are rational and $0 < \xi \le \eta$, there exist coprime $M, N \in \mathbb{N}$ with $M > N$ such that $(\eta + 1)/\xi = M/N$. Plugging the latter into (4) yields

$$\frac{x}{z} = \xi = \frac{2MN + \nu N^2}{M^2 + \nu MN + N^2}, \quad \frac{y}{z} = \eta = \frac{M^2 - N^2}{M^2 + \nu MN + N^2}; \tag{5}$$

note that $M^2 + \nu MN + N^2 > 0$. From (5) and $x \le y$, it follows that

$$(x, y, z) = \frac{z}{Z_\nu(M, N)}\big(X_\nu(M, N), Y_\nu(M, N), Z_\nu(M, N)\big), \tag{6}$$

where the functions $X_\nu, Y_\nu, Z_\nu : \mathbb{N}^2 \to \mathbb{Z}$ are

$$\begin{aligned} X_\nu(p, q) &= \min\{2pq + \nu q^2, p^2 - q^2\}, \\ Y_\nu(p, q) &= \max\{2pq + \nu q^2, p^2 - q^2\}, \\ Z_\nu(p, q) &= p^2 + \nu pq + q^2. \end{aligned}$$

The crux of the proof, then, is to show that (6) holds with (M, N) replaced by a unique pair (m, n) having the additional properties claimed in (i), (ii), and (iii), respectively, for which $X_\nu(m, n)$, $Y_\nu(m, n)$, and $Z_\nu(m, n)$ have no common factor—which, as seen above, implies that $z/Z_\nu(m, n) = 1$. To show all this, it is helpful to consider the possible values of ν separately.

Case I: $\nu = 0$**.** Denote by $\mathcal{N}_0$ the set of pairs specified by (i); that is, let

$$\mathcal{N}_0 = \{(p, q) \in \mathbb{N}^2 : \gcd(p, q) = 1, p - q \in \mathbb{N} \setminus 2\mathbb{N}\}.$$

Note that $X_0(p, q)$, $Y_0(p, q)$, and $Z_0(p, q)$ have no common factor for $(p, q) \in \mathcal{N}_0$. Indeed, if $2pq$ and $p^2 - q^2$ were divisible by a prime number t, then $t = 2$ because otherwise p and q would both be divisible by t, contradicting $\gcd(p, q) = 1$. Yet if $t = 2$, then $p - q \in 2\mathbb{N}$, which is not the case.

Now, if $(M, N) \in \mathcal{N}_0$, then simply take $(m, n) = (M, N)$. If, however, $(M, N) \notin \mathcal{N}_0$, that is, if $M - N \in 2\mathbb{N}$, then

$$X_0(M, N) = 2X_0(m, n), \quad Y_0(M, N) = 2Y_0(m, n), \quad Z_0(M, N) = 2Z_0(m, n),$$

where $m = \frac{1}{2}(M + N)$, $n = \frac{1}{2}(M - N)$, and, most importantly, $(m, n) \in \mathcal{N}_0$. Therefore, with (M, N) replaced by $(m, n) \in \mathcal{N}_0$, (6), in either case, holds for $\nu = 0$, $z/Z_0(m, n) = 1$, and hence (m, n) has all the properties claimed in (i).

It remains to show that $(m, n) \in \mathcal{N}_0$ in fact is unique. Assume (X_0, Y_0, Z_0) attains the same value for $(\widetilde{m}, \widetilde{n}) \in \mathcal{N}_0$. Then $\widetilde{m}^2 + \widetilde{n}^2 = m^2 + n^2$, and either $\widetilde{m}^2 - \widetilde{n}^2 = m^2 - n^2$ or $\widetilde{m}^2 - \widetilde{n}^2 = 2mn$. In the first case, $(\widetilde{m}^2, \widetilde{n}^2) = (m^2, n^2)$, hence $(\widetilde{m}, \widetilde{n}) = (m, n)$. In the second case, $2\widetilde{m}^2 = (m + n)^2$, which is impossible because 2 is not a square in $\mathbb{Q}$. This completes the proof of (i).

Case II: $\nu = 1$**.** Let $\mathcal{N}_1 = \{(p, q) \in \mathbb{N}^2 : \gcd(p, q) = 1, p - q \in \mathbb{N} \setminus 3\mathbb{N}\}$, i.e., $\mathcal{N}_1$ is the set of pairs specified by (ii). In analogy to Case I, $X_1(p, q)$, $Y_1(p, q)$, and $Z_1(p, q)$ have no common factor for $(p, q) \in \mathcal{N}_1$, and so simply let $(m, n) = (M, N)$ whenever $(M, N) \in \mathcal{N}_1$. If, however, $(M, N) \notin \mathcal{N}_1$, that is, if $M - N \in 3\mathbb{N}$, then

$$X_1(M, N) = 3X_1(m, n), \quad Y_1(M, N) = 3Y_1(m, n), \quad Z_1(M, N) = 3Z_1(m, n),$$

where $m = \frac{1}{3}(M + 2N)$, $n = \frac{1}{3}(M - N)$, and $(m, n) \in \mathcal{N}_1$. With (M, N) replaced by $(m, n) \in \mathcal{N}_1$, (6) holds for $\nu = 1$ in either case; hence, $z/Z_1(m, n) = 1$, and (m, n) has all the properties stipulated in (ii).

To show that $(m, n) \in \mathcal{N}_1$ again is unique, assume that (X_1, Y_1, Z_1) attains the same value for $(\widetilde{m}, \widetilde{n}) \in \mathcal{N}_1$. Then $\widetilde{m}^2 + \widetilde{m}\widetilde{n} + \widetilde{n}^2 = m^2 + mn + n^2$, and the pair $(2\widetilde{m}\widetilde{n} + \widetilde{n}^2, \widetilde{m}^2 - \widetilde{n}^2)$ equals either $(2mn + n^2, m^2 - n^2)$ or $(m^2 - n^2, 2mn + n^2)$. In the first case,

$$3\widetilde{m}\widetilde{n} = 2(2\widetilde{m}\widetilde{n} + \widetilde{n}^2) + (\widetilde{m}^2 - \widetilde{n}^2) - (\widetilde{m}^2 + \widetilde{m}\widetilde{n} + \widetilde{n}^2) = 3mn\,,$$

so $\widetilde{m}^2 + \widetilde{n}^2 = m^2 + n^2$, and hence $(\widetilde{m}^2, \widetilde{n}^2) = (m^2, n^2)$ and $(\widetilde{m}, \widetilde{n}) = (m, n)$. In the second case,

$$\begin{aligned} 3\widetilde{m}^2 &= -(2\widetilde{m}\widetilde{n} + \widetilde{n}^2) + (\widetilde{m}^2 - \widetilde{n}^2) + 2(\widetilde{m}^2 + \widetilde{m}\widetilde{n} + \widetilde{n}^2) \\ &= -(m^2 - n^2) + (2mn + n^2) + 2(m^2 + mn + n^2) = (m + 2n)^2\,, \end{aligned}$$

which is impossible because 3 is not a square in $\mathbb{Q}$. This proves (ii).

Case III: $\nu = -1$**.** Observe that (4) and $0 < \xi \le \eta$ imply $\xi \le 1$, $\eta \ge 1$, and

$$(\eta + 1)^2 - 4\xi^2 = 2 - 5\xi^2 + \xi\eta + 2\eta \ge 2 - 5\xi + \xi + 2\xi = 2(1 - \xi) \ge 0\,.$$

Hence, $(\eta + 1)/\xi = M/N \ge 2$, and in fact $M > 2N$ since otherwise $(\xi, \eta) = (1, 1)$ and $(x, y, z) = (1, 1, 1)$. Consequently, $X_{-1}(M, N) = 2MN - N^2$ and $Y_{-1}(M, N) = M^2 - N^2$. Let $\mathcal{N}_{-1} = \{(p, q) \in \mathbb{N}^2 : \gcd(p, q) = 1,\ p - 2q \in \mathbb{N} \setminus 3\mathbb{N}\}$ be the set of pairs in (iii). Similarly to the previous cases, it is easily checked that $X_{-1}(p, q)$, $Y_{-1}(p, q)$, and $Z_{-1}(p, q)$ have no common factor for $(p, q) \in \mathcal{N}_{-1}$ and so take $(m, n) = (M, N)$ whenever $(M, N) \in \mathcal{N}_{-1}$.

Next, define one more function $X^*_{-1} : \mathbb{N}^2 \to \mathbb{Z}$ by

$$X^*_{-1}(p, q) = |p^2 - 2pq| = Y_{-1}(p, q) - X_{-1}(p, q)\,. \tag{7}$$

Note that $X^*_{-1}(p, q)$, $Y_{-1}(p, q)$, and $Z_{-1}(p, q)$ have a common factor if and only if $X_{-1}(p, q)$, $Y_{-1}(p, q)$, and $Z_{-1}(p, q)$ do. Now, if $(M, N) \notin \mathcal{N}_{-1}$, that is, if $M - 2N \in 3\mathbb{N}$, then

$$\begin{aligned} &X_{-1}(M, N) = 3X^*_{-1}(m, n)\,, \quad Y_{-1}(M, N) = 3Y_{-1}(m, n)\,, \\ &Z_{-1}(M, N) = 3Z_{-1}(m, n)\,, \end{aligned}$$

where $m = \frac{1}{3}(2M - N)$, $n = \frac{1}{3}(M - 2N)$, and $(m, n) \in \mathcal{N}_{-1}$. Thus, with (M, N) replaced by $(m, n) \in \mathcal{N}_{-1}$ and with $X_{-1}(M, N)$ replaced by $X^*_{-1}(m, n)$ if $M - 2N \in 3\mathbb{N}$, (6) holds for $\nu = -1$ in either case, and (m, n) has all the properties claimed in (iii).

Finally, to establish uniqueness of (m, n), assume (Y_{-1}, Z_{-1}) attains the same value for $(\widetilde{m}, \widetilde{n}) \in \mathcal{N}_{-1}$. Thus, the values of Y_{-1}, Z_{-1}, and hence also $Z^2_{-1} - Y^2_{-1} = mn(3mn - 2Z_{-1})$ do not change when (m, n) is replaced by $(\widetilde{m}, \widetilde{n})$. Consequently, $\widetilde{m}\widetilde{n}(3\widetilde{m}\widetilde{n} - 2Z_{-1}) = Z^2_{-1} - Y^2_{-1}$, and solving this quadratic equation for $\widetilde{m}\widetilde{n}$ yields

$$3\widetilde{m}\widetilde{n} = Z_{-1} \pm \sqrt{4Z^2_{-1} - 3Y^2_{-1}} = m^2 - mn + n^2 \pm (m^2 - 4mn + n^2)\,.$$

It follows that either $3\widetilde{m}\widetilde{n} = 2m^2 - 5mn + 2n^2$ or $\widetilde{m}\widetilde{n} = mn$. In the first case,

$$3\widetilde{m}^2 = \tfrac{3}{2}(Y_{-1} + Z_{-1} + \widetilde{m}\widetilde{n}) = \tfrac{1}{2}(8m^2 - 8mn + 2n^2) = (2m-n)^2\,,$$

which is impossible. In the second case, $\widetilde{m}^2 + \widetilde{n}^2 = m^2 + n^2$, and so again $(\widetilde{m}, \widetilde{n}) = (m, n)$. This establishes (iii).

As predicted earlier, the converse now follows easily. By virtue of a very short calculation, all triples given by (i), (ii), and (iii), respectively, satisfy (3), hence are ordered ν-Pythagorean. That they are primitive also is clear from the fact that $X_\nu(m,n)$, $Y_\nu(m,n)$, and $Z_\nu(m,n)$ have no common factor for $(m,n) \in \mathcal{N}_\nu$. ■

The provision $(x, y, z) \neq (1, 1, 1)$ in Theorem 8 is relevant only for alternative (iii) since $(1, 1, 1)$ indeed is an ordered primitive (-1)-Pythagorean triple, formally obtained by choosing the (forbidden) value $(1, 0)$ for (m, n). Also, if (x, y, z) with $x \geq 0$ is an ordered primitive solution of (3) for $\nu = -1$, then so is $(y - x, y, z)$. This motivates (7) and explains why in (iii) there are two possible values for x, unlike in (i) and (ii). Figure 3 lists the five (lexicographically) smallest triples of each type.

Pythagorean	pseudo-Pythagorean	
$\nu = 0$	$\nu = 1$	$\nu = -1$
(3, 4, 5)	(3, 5, 7)	(1, 1, 1)
(5, 12, 13)	(5, 16, 19)	(3, 8, 7)
(7, 24, 25)	(7, 8, 13)	(5, 8, 7)
(8, 15, 17)	(7, 33, 37)	(5, 21, 19)
(9, 40, 41)	(9, 56, 61)	(7, 15, 13)

Figure 3. The first five ordered primitive ν-Pythagorean triples; see Theorem 8.

5. WHICH SIMPLE TRIANGLES ARE RATIONAL? Recall that a triangle $\Delta = \langle \delta_1, \delta_2, \delta_3 \rangle$ is *rational* if each angle δ_j is rational. The following well-known consequence of Theorem 7 has already been mentioned in the Introduction: An elementary triangle is rational (if and) only if it is equilateral. But for one trivial exception, therefore, being elementary and being rational are mutually exclusive triangle properties. Not only from a geometry student's perspective, it is natural to look for a slightly larger class of simple triangles that contains more than just one rational triangle. This final section briefly discusses two such classes.

Call a triangle *almost elementary* if at least two of the positive real numbers u_j in $\ell_1 : \ell_2 : \ell_3 = u_1 : u_2 : u_3$ are integers. Trivially, every elementary triangle is almost elementary, but the converse is not true in general, e.g., take $\Delta_4 = \langle 45, 45, 90 \rangle$ with $\ell_1 : \ell_2 : \ell_3 = 1 : 1 : \sqrt{2}$. Note that Δ_4 is both rational and isosceles. Every isosceles triangle clearly is almost elementary. The class of almost elementary triangles therefore contains infinitely many rational triangles. As asserted by Claim 2, but for one exception, no other rational almost elementary triangles exist. The following is a more precise version of Claim 2.

Theorem 9. *Let $\Delta = \langle \delta_1, \delta_2, \delta_3 \rangle$ be an almost elementary triangle. If Δ is rational, then exactly one of the following alternatives applies:*

(i) *There exists a unique* $\delta \in \mathbb{Q}$ *with* $0 < \delta < 90$ *such that*

$$\Delta = \langle \min\{\delta, 180 - 2\delta\}, \delta, \max\{\delta, 180 - 2\delta\} \rangle ,$$

i.e., Δ *is isosceles;*

(ii) $\Delta = \langle 30, 60, 90 \rangle$.

Proof. Let $\Delta = \langle \delta_1, \delta_2, \delta_3 \rangle$ be a rational almost elementary triangle. Hence, there exist $j \neq k$ such that $\ell_j/\ell_k \in \mathbb{Q}$. Recall that, by the law of sines, $\ell_j/\ell_k = \sin \delta_j^\circ / \sin \delta_k^\circ$. Thus, $\cos(\delta_j - 90)^\circ / \cos(\delta_k - 90)^\circ$ is rational, and $\delta_j, \delta_k \in \mathbb{Q}$ by assumption. Without loss of generality, assume $\delta_j \geq \delta_k$, and consequently, $\delta_k < 90$. Suppose first that $\delta_j > 90$. Then, by Corollary 5, $\delta_j - 90 = 90 - \delta_k$, which is impossible. Next consider the case $\delta_j = 90$. Then $\cos(90 - \delta_k)^\circ \in \mathbb{Q}$, and $90 - \delta_k = 60$, by Corollary 4. Thus, $\Delta = \langle 30, 60, 90 \rangle$, and this indeed is an almost elementary triangle with $\ell_1 : \ell_2 : \ell_3 = 1 : \sqrt{3} : 2$. Finally, if $\delta_j < 90$, then Corollary 5 yields $\delta_j = \delta_k$, and so, with the appropriate $\delta \in \mathbb{Q}$ either $\Delta = \langle \delta, \delta, 180 - 2\delta \rangle$ and $0 < \delta \leq 60$, or else $\Delta = \langle 180 - 2\delta, \delta, \delta \rangle$ and $60 < \delta < 90$. ■

$\langle 15, 15, 150 \rangle$	$\sqrt{2} : \sqrt{2} : \sqrt{3} + 1$
$\langle 30, 30, 120 \rangle$	$1 : 1 : \sqrt{3}$
$\langle 30, 75, 75 \rangle$	$\sqrt{3} - 1 : \sqrt{2} : \sqrt{2}$
$\langle 36, 36, 108 \rangle$	$2 : 2 : \sqrt{5} + 1$
$\langle 36, 72, 72 \rangle$	$\sqrt{5} - 1 : 2 : 2$
$\langle 45, 45, 90 \rangle$	$1 : 1 : \sqrt{2}$
$\langle 60, 60, 60 \rangle$	$1 : 1 : 1$

$\langle 15, 30, 135 \rangle$	$\sqrt{3} - 1 : \sqrt{2} : 2$
$\langle 15, 45, 120 \rangle$	$\sqrt{3} - 1 : 2 : \sqrt{6}$
$\langle 15, 60, 105 \rangle$	$\sqrt{3} - 1 : \sqrt{6} : \sqrt{3} + 1$
$\langle 15, 75, 90 \rangle$	$\sqrt{3} - 1 : \sqrt{3} + 1 : 2\sqrt{2}$
$\langle 30, 45, 105 \rangle$	$\sqrt{2} : 2 : \sqrt{3} + 1$
$\langle 30, 60, 90 \rangle$	$1 : \sqrt{3} : 2$
$\langle 45, 60, 75 \rangle$	$2 : \sqrt{6} : \sqrt{3} + 1$

Figure 4. There exist exactly 14 rational grade school triangles, of which seven are isosceles (left table), three are right (gray boxes; see [**3**]), and five are neither; see [**1, 14**] for details. Note that, while eight triangles are almost elementary, only the equilateral is elementary.

Another interesting enlargement of the class of elementary triangles, implicitly suggested by [**3**], is as follows. Call a triangle a *grade school* triangle if each number u_j in $\ell_1 : \ell_2 : \ell_3 = u_1 : u_2 : u_3$ is an element of a real quadratic number field, i.e., $u_j \in \mathbb{Q}(\sqrt{d_j})$ for some squarefree integer $d_j \geq 2$. (Recall that an integer is *squarefree* if it is not divisible by p^2 for any prime number p.) By scaling (i.e., by choosing an appropriate unit length), it can be assumed that $u_j = k_j + l_j\sqrt{d_j}$ with $k_j, l_j \in \mathbb{Z}$. Again, it is clear that every elementary triangle is grade school, whereas the converse is not true in general, as the examples of Δ_4 and $\Delta_5 = \langle 30, 45, 105 \rangle$ show; for the latter $\ell_1 : \ell_2 : \ell_3 = \sqrt{2} : 2 : \sqrt{3} + 1$. As seen earlier, Δ_4 is almost elementary, whereas Δ_5 clearly is not. On the other hand, $\Delta_6 = \langle 54, 54, 72 \rangle$ is a (rational) almost elementary triangle that is not grade school since $\ell_1 : \ell_2 : \ell_3 = \sqrt{2} : \sqrt{2} : \sqrt{5 - \sqrt{5}}$. Thus, the almost elementary and grade school properties in general are unrelated. Also note that Δ_4 and Δ_5 both are rational, which shows that rational grade school triangles other than the equilateral do exist. Unlike in the almost elementary case, however, there is no infinite supply of rational grade school triangles. In fact, only very few such triangles exist. The following *classification theorem of rational grade school triangles* makes this assertion precise. The result is stated here to give an impression of how scarce rational grade school triangles are; for details, the interested reader is referred to [**1**], which only uses the irreducibility of certain polynomials with integer coefficients, and to [**14**], which gives an alternative account employing basic Galois theory.

Theorem 10. *Let $\Delta = \langle \delta_1, \delta_2, \delta_3 \rangle$ be a grade school triangle. Then the following are equivalent:*

(i) Δ *is rational;*

(ii) *Each number* δ_1, δ_2, δ_3 *is an integer multiple of* 15 *or* 36*;*

(iii) Δ *equals exactly one of the* 14 *triangles listed in* Figure 4.

Finally, it may be worthwhile to relate the observations in this article to another classical concept in geometry, namely the constructibility by compass and straightedge [**7**, Nr. 35–37]. Thus, call a triangle *constructible* if it has a realization that is constructible by compass and straightedge. Since $\sqrt{d}$ is constructible for every $d \in \mathbb{N}$, every grade school triangle, and a fortiori every elementary triangle, is constructible. By contrast, even a rational almost elementary triangle need not be constructible, as the example of $\Delta_7 = \langle 40, 70, 70 \rangle$ shows. Figure 5 schematically depicts the relations

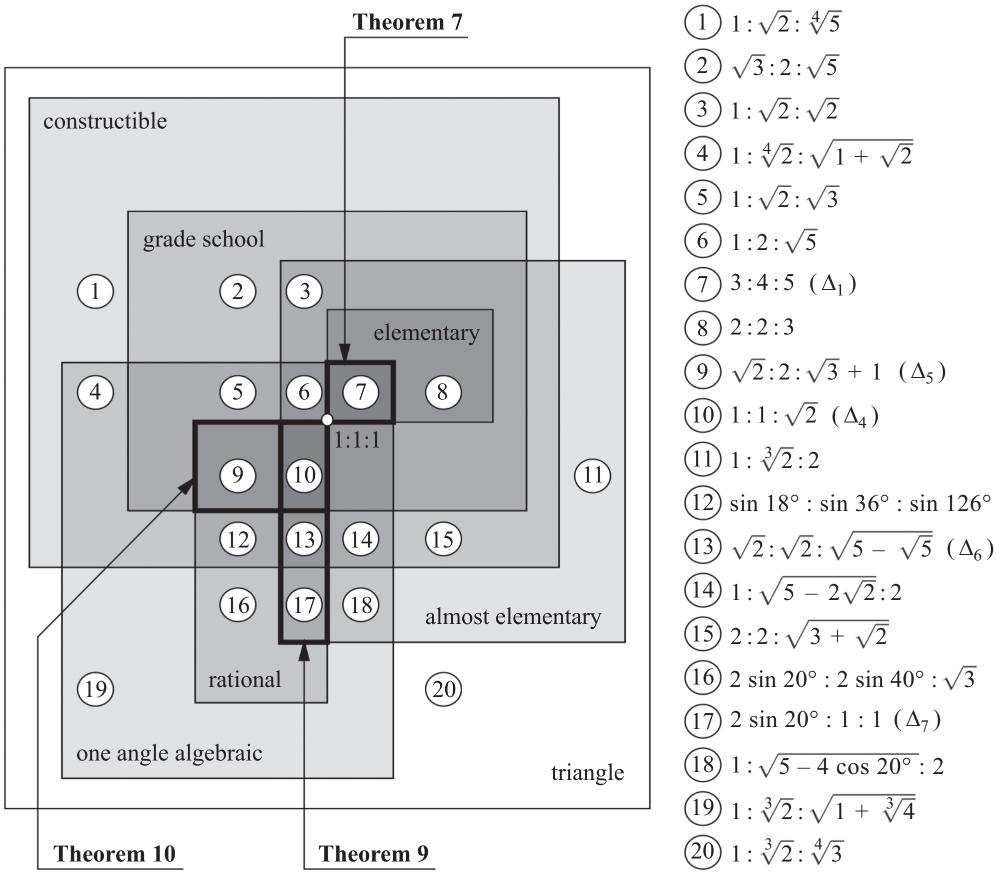

Figure 5. Illustrating the relations between the properties of (similarity types of planar) triangles discussed in this article; an example of $\ell_1 : \ell_2 : \ell_3$ is given for each situation.

between the concepts discussed in this article and illustrates each possible situation through a concrete example.

ACKNOWLEDGMENTS. The author was supported by an NSERC Discovery Grant. He is much indebted to S. Gille and A. Weiss for helpful discussions and comments. A big *thank you* to J. Calcut and G. Molteni for pointing out several important references. The thoughtful, repeated suggestions made by an anonymous referee greatly helped improve the exposition.

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ARNO BERGER is a professor of mathematics at the University of Alberta. Before settling in Canada, he held academic positions in Austria and New Zealand and spent extended research visits to the U.S., U.K., and Germany. A native of Austria, he received his Ph.D. from the Vienna University of Technology in 1997.
Mathematical and Statistical Sciences, University of Alberta, Edmonton AB T6G 2G1, Canada
berger@ualberta.ca