NONAUTONOMOUS FINITE-TIME DYNAMICS

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ABSTRACT. Nonautonomous differential equations on finite-time intervals play
an increasingly important role in applications that incorporate time-varying
vector fields, e.g. observed or forecasted velocity fields in meteorology or oceano-
graphy which are known only for times t from a compact interval. While clas-
sical dynamical systems methods often study the behaviour of solutions as
$t \to \pm\infty$, the dynamic partition (originally called the EPH partition) aims at
describing and classifying the finite-time behaviour. We discuss fundamental
properties of the dynamic partition and show that it locally approximates the
nonlinear behaviour. We also provide an algorithm for practical computations
with dynamic partitions and apply it to a nonlinear 3-dimensional example.

1. **Introduction.** In this article, we study nonautonomous ordinary differential
equations

$$\dot{x} = f(t, x), \tag{1}$$

where $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous with continuous derivatives $D_t f$, $D_x^2 f$, and
$I \subset \mathbb{R}$ is a nonempty interval. We are particularly interested in the finite-time case,
i.e., in the case of a compact interval $I = [t_0, t_1]$ with $t_0 < t_1$. *Finite-time dynamics*
has recently become a very active field of research, see e.g. [6, 7, 10, 11, 12, 13, 14,
16, 17, 18, 19, 20, 21, 22] and the many references therein. Some of the reasons for
the increased interest in finite-time dynamics are:

(i) *Numerically computed* vector fields are of the form (1). Modern developments
   in scientific computing, e.g. in computational fluid dynamics, often yield a
   discretized version of a time-varying velocity field, i.e. an equation of the form
   (1) where the numerical simulation starts at time t_0 and ends at t_1.

(ii) *Observed* velocity fields are of the form (1). Recent advances in technology, e.g. in satellite imaging of ocean currents, enable the generation of high-resolution data sets on a space-time grid, i.e. a system of the form (1) where the observation of the vector field f starts at time t_0 and ends at t_1.

(iii) The *transient behaviour* of (1) can be observed on finite-time intervals. Typically, the qualitative behaviour of solutions of (1) as $t \to \pm\infty$ is independent of the transient behaviour of solutions on any finite-time interval of the form $[t_0, t_1]$ since the parts $f|_{(-\infty,t_0]}(\cdot,x)$, $f|_{[t_0,t_1]}(\cdot,x)$ and $f|_{[t_1,\infty)}(\cdot,x)$ may not be correlated at all. As a consequence, if (1) does not exhibit a simple time dependence (e.g., almost periodicity) then the transient behaviour for $t \in [t_0, t_1]$ often is physically much more relevant than the asymptotic behaviour as $t \to \pm\infty$.

For a concrete application to 2-dimensional fluid dynamics consider two like-signed vortices of the same size which are close together, circle around each other and after a while merge to a single vortex [23]. By changing to a co-rotating coordinate system the average rotation of the system can be factored out, see Figure 1 for a snapshot of the vector fields before and after merging [5]. During the process
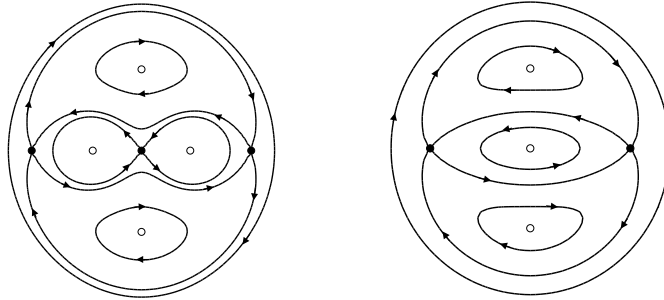

FIGURE 1. Schematic shape of the velocity field of two 2-dimensional vortices in a co-rotating frame before (left) and after merging.

of merging the dynamics in the center changes from a saddle-type structure to a circular structure. Since the merging process takes place in finite time, asymptotic tools are not applicable. Rather, genuine finite-time concepts are required to analyze this situation. It is the purpose of this article to develop in some generality one such concept, the *dynamic partition* (originally referred to as the EHP partition), and to discuss its potential applications as well as inherent limitations.

Which concepts are adequate for studying the qualitative behaviour of (1) on finite-time intervals? A fundamental problem consists in developing a proper notion of *hyperbolicity* for the linear system

$$\dot{x} = A(t)x, \tag{2}$$

where $A : I \to \mathbb{R}^{n \times n}$ is continuous. Recall that, for $I = \mathbb{R}$, (2) is termed *hyperbolic* if it admits an *exponential dichotomy*, i.e., if there exists an invariant projector $P : I \to \mathbb{R}^n$ and constants $\alpha > 0$, $K \geq 1$ such that for all $s, t \in I$ and $\xi \in \mathbb{R}^n$,

$$\begin{aligned} \|\Phi(t,s)P(s)\xi\| &\leq Ke^{-\alpha(t-s)}\|\xi\| && \text{if } t \geq s, \\ \|\Phi(t,s)[\mathrm{id} - P(s)]\xi\| &\leq Ke^{\alpha(t-s)}\|\xi\| && \text{if } t \leq s; \end{aligned} \tag{3}$$

here Φ denotes the evolution operator associated with (2). The two vector bundles $S = \{(t, x) : t \in I, x \in \operatorname{im} P(t)\}$ and $U = \{(t, x) : t \in I, x \in \ker P(t)\}$ are invariant and consist of solutions which, respectively, decay and grow exponentially with a uniform rate at least α. Thus if for instance $P \equiv \operatorname{id}$ then (2) is exponentially stable. Transient behaviour is not captured by this notion as the constant K can be very large. On the other hand, (3) with $K = 1$ implies that the norm of solutions in S and U, respectively, decays and increases monotonically for $t \in I$. While exponential decay and growth on $I = \mathbb{R}$ is independent of the chosen norm, monotonicity of solutions in S and U is not: the transient behaviour of (1) or (2) will be different depending on which norm on $\mathbb{R}^n$ is used. Not least from a practical point of view, therefore, is it imperative to allow for more than one norm. This is one reason why we pay considerable attention to the family of norms $\| \cdot \|_\Gamma = \sqrt{\langle \cdot, \Gamma \cdot \rangle}$ induced by different symmetric, positive definite matrices Γ. Another reason is that the invariant projector P in (3) for $I = [t_0, t_1]$ is generally not unique. A careful choice of Γ, however, can yield uniqueness [2] and thus in turn make accessible deeper structure theorems for finite-time dynamics. These further developments evidently rely on the theory of dynamic partitions developed here.

In this article, we follow and extend the approach by Haller [7, 10], see also [4, 5]. We study the behaviour of solutions of (1) in the vicinity of a particular solution $\mu : I \to \mathbb{R}^n$. By the transformation $x \mapsto x - \mu$ the solution μ is mapped to the zero solution of

$$\dot{x} = D_x f\big(t, \mu(t)\big) x + g(t, x), \tag{4}$$

with the nonlinearity $g(t, x) := f\big(t, x + \mu(t)\big) - f(t, \mu(t)) - D_x f\big(t, \mu(t)\big) x$. We denote the linearization of (1) along μ, which clearly also is the linearization of (4) at $x = 0$, by

$$\dot{\xi} = D_x f\big(t, \mu(t)\big) \xi. \tag{5}$$

The local behaviour of (1) near μ is described by (5), and we partition the extended state space $I \times \mathbb{R}^n$ into attracting, repelling, hyperbolic, quasihyperbolic, elliptic and degenerate points, according to their local (in x and t) behaviour. Though similar in spirit to [4, 7, 10] our approach here is neither restricted to planar systems nor to the usage of the Euclidean norm. While largely extending the applicability of dynamic partitions, this increase in generality turns out to cause only minor mathematical difficulties.

This article is organized as follows. In Section 2.1 we introduce the dynamic partition and discuss its invariance under orthogonal transformations and translations (Lemma 2.5). An arbitrary linear transformation maps the dynamic partition to a transformed dynamic partition, while at the same time the norm in question is also altered (Lemma 2.7). It is shown that all but the degenerate part of the dynamic partition are open sets in the extended state space (Lemma 2.8). For linear systems with constant coefficients the finite-time notions are related to the classical notions of attraction, repulsion, hyperbolicity and ellipticity (Theorem 2.9). An application to autonomous equations yields a result on the location of periodic orbits (Theorem 2.10). In Section 2.2 we show that even though it is defined in terms of the linearization (5), the dynamic partition locally approximates the nonlinear behaviour of (1) (Theorems 2.11, 2.13, 2.14). Section 2.3 is dedicated to the presentation of an algorithm to practically compute dynamic partitions. This algorithm is applied to an example in Section 2.4.

2. **Finite-time dynamics.** Throughout this article, $\Gamma \in \mathbb{R}^{n \times n}$ denotes a symmetric positive definite matrix, that is, $\Gamma = \Gamma^\top > 0$, and $\|\cdot\|_\Gamma$ symbolizes the induced norm, i.e. $\|x\|_\Gamma = \sqrt{\langle x, \Gamma x \rangle}$ for all $x \in \mathbb{R}^n$. Quantities depending on Γ have their dependence made explicit by a subscript which is suppressed only if Γ equals $\mathrm{id}_{n \times n}$, the $n \times n$ identity matrix.

2.1. **Dynamic partition.** For an arbitrary solution $\xi : I \to \mathbb{R}^n$ of (5) the instantaneous change of $\frac{1}{2}\|\xi\|_\Gamma^2$ is given by

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\xi(t)\|_\Gamma^2 &= \frac{1}{2}\frac{d}{dt}\langle \xi(t), \Gamma \xi(t) \rangle \\
&= \left\langle \xi(t), \frac{1}{2}\left[\Gamma A\big(t, \mu(t)\big) + A\big(t, \mu(t)\big)^\top \Gamma\right]\xi(t) \right\rangle \quad (6)
\end{aligned}
$$

with $A(t, x) = D_x f(t, x)$.

Definition 2.1 (Γ-strain tensor). The symmetric matrix

$$
S_\Gamma(t, x) := \frac{1}{2}\left[\Gamma A(t, x) + A(t, x)^\top \Gamma\right]
$$

is called the Γ-*strain tensor* of equation (1).

Thus the Γ-strain tensor describes growth and decay of solutions ξ of the linearization (5) with respect to the norm $\|\cdot\|_\Gamma$. For $\Gamma = \mathrm{id}_{n \times n}$, the matrix $S = S_{\mathrm{id}_{n \times n}}$ is called *(rate-of-) strain tensor*. Clearly, all solutions of (5) are strictly decreasing on $I \times \mathbb{R}^n$ w.r.t. the $\|\cdot\|_\Gamma$ norm if $S_\Gamma\big(t, \mu(t)\big)$ is negative definite, i.e., if for all $t \in I$

$$
\langle \xi, S_\Gamma\big(t, \mu(t)\big)\xi \rangle < 0 \qquad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}.
$$

Next we define the set of zero strain, see also [7].

Definition 2.2 (Zero Γ-Strain Set). The set

$$
Z_\Gamma(t, x) := \{\xi \in \mathbb{R}^n : \langle \xi, S_\Gamma(t, x)\xi \rangle = 0\}
$$

is called the *zero Γ-strain set* of equation (1).

For any symmetric matrix $L \in \mathbb{R}^{n \times n}$ let $\lambda_1, \ldots, \lambda_n$ denote its eigenvalues, ordered according to

$$
\lambda_1 \geq \ldots \geq \lambda_{n^+} > 0, \quad \lambda_{n^+ + 1}, \ldots, \lambda_{n-n^-} = 0, \quad 0 > \lambda_{n-n^- + 1} \geq \ldots \geq \lambda_n, \quad (7)
$$

for some $n^+, n^- \in \{0, 1, \ldots, n\}$ with $n^+ + n^- \leq n$. We refer to (n^+, n^-) as the *type* of L, and L is *degenerate* if $n^+ + n^- < n$ or, equivalently, if $\det L = 0$; otherwise L is *non-degenerate*. Clearly, L is positive definite if and only if it is of type $(n, 0)$, and it is negative definite precisely if it is of type $(0, n)$. If L is non-degenerate then it is indefinite if and only if it has eigenvalues of different sign, i.e. $n^+ n^- \neq 0$.

Proposition 1 (Characterization of Zero Γ-Strain Set). *Consider system* (1) *with Γ-strain tensor S_Γ. Then the zero strain set Z_Γ satisfies:*

(i) Z_Γ is the origin if and only if S_Γ is positive or negative definite.

(ii) Z_Γ is a cone if and only if S_Γ is indefinite or degenerate.

Proof. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of S_Γ, ordered as in (7), and $Q \in \mathbb{R}^{n \times n}$ an orthogonal matrix such that

$$
Q^\top S_\Gamma Q = \mathrm{diag}(\lambda_1, \ldots, \lambda_{n^+}, 0, \ldots, 0, \lambda_{n-n^- + 1}, \ldots, \lambda_n).
$$

Then $Z_\Gamma = \{Q\xi \in \mathbb{R}^n \,:\, \langle \xi, Q^\top S_\Gamma Q\xi \rangle = 0\}$ and therefore

$$Z_\Gamma = Q\{\xi \in \mathbb{R}^n \,:\, \lambda_1 \xi_1^2 + \ldots + \lambda_{n^+}\xi_{n^+}^2 + \lambda_{n-n^-+1}\xi_{n-n^-+1}^2 + \ldots + \lambda_n \xi_n^2 = 0\}\,, \quad (8)$$

and the claim follows. $\qquad\square$

Remark 1. If S_Γ is degenerate then Z_Γ contains the $(n - n^+ - n^-)$-dimensional subspace

$$Q\{(0,\ldots,0,\xi_{n^++1},\ldots,\xi_{n-n^-},0,\ldots,0)^\top \in \mathbb{R}^n \,:\, \xi_j \in \mathbb{R}\}\,,$$

with Q as above. For example, Z_Γ is a line in $\mathbb{R}^3$ if $Q^\top S_\Gamma Q = \mathrm{diag}(1,1,0)$, or a union of two planes if $Q^\top S_\Gamma Q = \mathrm{diag}(1,0,-1)$.
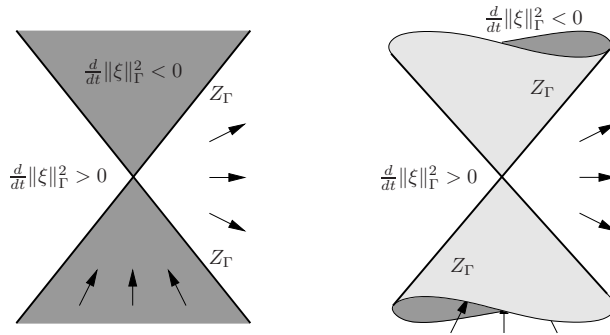

FIGURE 2. $Z_\Gamma = Z_\Gamma\big(t_0, \mu(t_0)\big)$ for equation (5) and fixed $t_0 \in I$ when Z_Γ is a cone for $n = 2$ (left) and $n = 3$ (right). Arrows and shading indicate the instantaneous norm change at $t = t_0$ for solutions of (5) which are not in Z_Γ.

Remark 2. If S_Γ is indefinite and non-degenerate then (8) reads

$$Z_\Gamma = Q\{\xi \in \mathbb{R}^n \,:\, \lambda_1 \xi_1^2 + \ldots + \lambda_{n^+}\xi_{n^+}^2 = -\lambda_{n^++1}\xi_{n^++1}^2 - \ldots - \lambda_n \xi_n^2\}$$

with $n^+ + n^- = n$ and $n^+ n^- > 0$, and replacing ξ_i by $\xi_i/\sqrt{|\lambda_i|}$ for all $i = 1,\ldots,n$ yields

$$Z_\Gamma = Q\left\{\left(\frac{\xi_1}{\sqrt{|\lambda_1|}},\ldots,\frac{\xi_n}{\sqrt{|\lambda_n|}}\right)^\top \in \mathbb{R}^n \,:\, \xi_1^2 + \ldots + \xi_{n^+}^2 = \xi_{n^++1}^2 + \ldots + \xi_n^2\right\}.$$

If, for some $t_0 \in I$, $S_\Gamma\big(t_0, \mu(t_0)\big)$ is indefinite and non-degenerate then $Z_\Gamma\big(t_0, \mu(t_0)\big)$ is a cone, and if the solution ξ of (5) crosses that cone, i.e., if $\xi(t_0) \in Z_\Gamma\big(t_0, \mu(t_0)\big)$, then the sign of the second derivative of $t \mapsto \|\xi(t)\|_\Gamma^2$ at $t = t_0$ characterizes whether ξ crosses transversally from a region with increasing to a region with decreasing norm or vice versa. With $S_\Gamma(t) = S_\Gamma\big(t, \mu(t)\big)$ and $\dot{S}_\Gamma(t) = \frac{d}{dt}S_\Gamma\big(t, \mu(t)\big)$ we find

$$\begin{aligned}
\frac{1}{2}\frac{d^2}{dt^2}\|\xi(t)\|_\Gamma^2 &= \frac{d}{dt}\langle \xi(t), S_\Gamma(t)\xi(t)\rangle \\
&= \langle \dot{\xi}(t), S_\Gamma(t)\xi(t)\rangle + \langle \xi(t), \dot{S}_\Gamma(t)\xi(t) + S_\Gamma(t)\dot{\xi}(t)\rangle \\
&= \langle A(t)\xi(t), S_\Gamma(t)\xi(t)\rangle + \langle \xi(t), \dot{S}_\Gamma(t)\xi(t) + S_\Gamma(t)A(t)\xi(t)\rangle \\
&= \langle \xi(t), [\dot{S}_\Gamma(t) + S_\Gamma(t)A(t) + A(t)^\top S_\Gamma(t)]\xi(t)\rangle\,.
\end{aligned}$$

Definition 2.3 (Γ-Strain Acceleration Tensor)**.** The symmetric matrix

$$M_\Gamma(t,x) := \dot{S}_\Gamma(t,x) + S_\Gamma(t,x)A(t,x) + A(t,x)^\top S_\Gamma(t,x)$$

is called the Γ-*strain acceleration tensor* of equation (1), where the symmetric matrix $\dot{S}_\Gamma(t,x) := D_t S_\Gamma(t,x) + D_x S_\Gamma(t,x)f(t,x)$ is the derivative of the Γ-symmetric part $S_\Gamma(t,x)$ of $A(t,x) = D_x f(t,x)$ along the solution of (1) starting at time t in x; note that $\dot{S}_\Gamma(t) = \dot{S}_\Gamma(t,\mu(t))$.
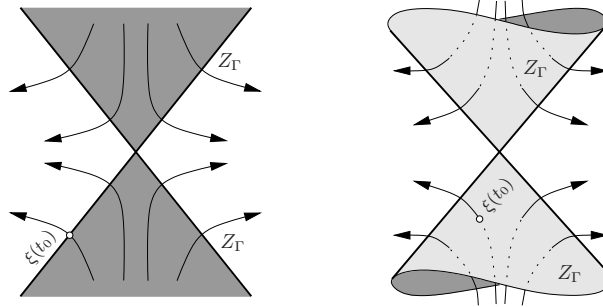

FIGURE 3. Local behaviour of solutions ξ of (5) with $\xi(t_0) \in Z_\Gamma = Z_\Gamma(t_0, \mu(t_0))$ and $\langle \xi(t_0), M_\Gamma(t_0, \mu(t_0))\xi(t_0)\rangle > 0$ when Z_Γ is a cone for $n = 2$ (left) and $n = 3$ (right).

The restriction of the quadratic form $\xi \mapsto \langle \xi, M_\Gamma(t,x)\xi\rangle$ to $Z_\Gamma(t,x)$ is denoted by $M_{Z_\Gamma}(t,x)$. Given $(t,x) \in I \times \mathbb{R}^n$ it is customary to call M_{Z_Γ} negative/positive definite if it attains only negative/positive values for all $\xi \in Z_\Gamma(t,x)\backslash\{0\}$, and indefinite if it attains both negative and positive values on $\mathbb{Z}_\Gamma(t,x)$. The following definition extends the EPH partition introduced by Haller [7].

Definition 2.4 (Dynamic Partition)**.** For the differential equation (1) and $\Gamma = \Gamma^\top > 0$ we define the following subsets of $I \times \mathbb{R}^n$:

(i) *Attracting region:*

$$\mathcal{A}_\Gamma := \{(t,x) \in I \times \mathbb{R}^n : S_\Gamma(t,x) \text{ is negative definite}\}$$

(ii) *Repelling region:*

$$\mathcal{R}_\Gamma := \{(t,x) \in I \times \mathbb{R}^n : S_\Gamma(t,x) \text{ is positive definite}\}$$

(iii) *Elliptic region:*

$$\mathcal{E}_\Gamma := \left\{(t,x) \in I \times \mathbb{R}^n : \begin{array}{l} S_\Gamma(t,x) \text{ is indefinite and non-degenerate,} \\ \text{and } M_{Z_\Gamma}(t,x) \text{ is indefinite} \end{array}\right\}$$

(iv) *Hyperbolic region:*

$$\mathcal{H}_\Gamma := \left\{(t,x) \in I \times \mathbb{R}^n : \begin{array}{l} S_\Gamma(t,x) \text{ is indefinite and non-degenerate,} \\ \text{and } M_{Z_\Gamma}(t,x) \text{ is positive definite} \end{array}\right\}$$

(v) *Quasihyperbolic region:*

$$\mathcal{Q}_\Gamma := \left\{(t,x) \in I \times \mathbb{R}^n : \begin{array}{l} S_\Gamma(t,x) \text{ is indefinite and non-degenerate,} \\ \text{and } M_{Z_\Gamma}(t,x) \text{ is negative definite} \end{array}\right\}$$

(vi) *Degenerate region:*

$$\mathcal{D}_\Gamma := (I \times \mathbb{R}^n) \setminus [\mathcal{A}_\Gamma \cup \mathcal{R}_\Gamma \cup \mathcal{E}_\Gamma \cup \mathcal{H}_\Gamma \cup \mathcal{Q}_\Gamma]$$

We say that $(t, x) \in I \times \mathbb{R}^n$ is of *type* $\mathcal{T}_\Gamma$ for $\mathcal{T} \in \{\mathcal{A}, \mathcal{R}, \mathcal{E}, \mathcal{H}, \mathcal{Q}, \mathcal{D}\}$ if $(t, x) \in \mathcal{T}_\Gamma$, or equivalently, if x is contained in the *t-fiber* $\mathcal{T}_\Gamma(t) = \{x \in \mathbb{R}^n : (t, x) \in \mathcal{T}_\Gamma\}$. As always, the subscript Γ is suppressed if $\Gamma = \mathrm{id}_{n \times n}$, i.e., we write $\mathcal{T}$ instead of $\mathcal{T}_{\mathrm{id}_{n \times n}}$.

Remark 3. If $\mu(t) \in \mathcal{H}_\Gamma(t)$ for all $t \in I$ then the cone field

$$\left\{ (t, \xi) \in I \times \mathbb{R}^n : \left\langle \xi, S_\Gamma\big(t, \mu(t)\big)\xi \right\rangle > 0 \right\}$$

is forward invariant under the dynamics of the linearization (5), i.e., $\frac{d}{dt}\|\xi(t_1)\|_\Gamma > 0$ implies $\frac{d}{dt}\|\xi(t_2)\|_\Gamma > 0$ for all $t_2 \in I$, $t_2 \geq t_1$. Similarly, if $\mu(t) \in \mathcal{Q}_\Gamma(t)$ for all $t \in I$ then $\left\{ (t, \xi) \in I \times \mathbb{R}^n : \left\langle \xi, S_\Gamma\big(t, \mu(t)\big)\xi \right\rangle < 0 \right\}$ is forward invariant.

Remark 4. The linearization of a linear system $\dot{x} = A(t)x$ along an arbitrary solution μ is the linear system itself. Hence, for every $t \in I$, the fiber $\mathcal{T}_\Gamma(t)$ with $\mathcal{T} \in \{\mathcal{A}, \mathcal{R}, \mathcal{E}, \mathcal{H}, \mathcal{Q}, \mathcal{D}\}$ is either empty or else equals $\mathbb{R}^n$. In this situation, we say that $\dot{x} = A(t)x$ is attracting/repelling/elliptic/hyperbolic/quasihyperbolic/degenerate at time $t \in I$ if the corresponding fiber $\mathcal{T}_\Gamma(t)$ is $\mathbb{R}^n$.

Remark 5. If (1) is autonomous, i.e., if $\dot{x} = F(x)$, then S_Γ, Z_Γ and M_Γ do not depend on t. As a consequence, each fiber $\mathcal{T}_\Gamma \equiv \mathcal{T}_\Gamma(t)$ for $\mathcal{T} \in \{\mathcal{A}, \mathcal{R}, \mathcal{E}, \mathcal{H}, \mathcal{Q}, \mathcal{D}\}$ is independent of t. In particular, a linear autonomous system $\dot{x} = Ax$ has one type $\mathcal{T}_\Gamma$, i.e., it is either attracting, repelling, elliptic, hyperbolic, quasihyperbolic or degenerate.

Remark 6. If (t, x) is hyperbolic or quasihyperbolic then

$$|\langle \xi, M_\Gamma(t, x)\xi \rangle| > 0 \qquad \text{for all } \xi \in Z_\Gamma(t, x) \setminus \{0\}\,,$$

whereas (t, x) is elliptic precisely if there exist vectors $\xi, \eta \in Z_\Gamma(t, x)$ with

$$\langle \xi, M_\Gamma(t, x)\xi \rangle > 0 > \langle \eta, M_\Gamma(t, x)\eta \rangle\,.$$

However, the set $\{\xi \in Z_\Gamma(t, x) : \langle \xi, M_\Gamma(t, x)\xi \rangle = 0\}$, being the intersection of the zero sets of two quadratic forms, can be quite complicated in the elliptic case.

Remark 7. If the norm $\|\cdot\|_{\Gamma(t)} = \sqrt{\langle \cdot, \Gamma(t)\cdot \rangle}$ is allowed to depend on time $t \in I$ then the strain tensor $S_\Gamma(t, x) = \frac{1}{2}[A(t, x)\Gamma(t) + \Gamma(t)A(t, x)^\top + \dot{\Gamma}(t)]$ depends on the derivative of $t \mapsto \Gamma(t)$ and one can easily choose $\Gamma(\cdot)$ such that $(t, x) \in \mathcal{A}_\Gamma$ or $(t, x) \in \mathcal{R}_\Gamma$. A full description of the relation between the dynamic partition and time-dependent norms is beyond the scope of this paper.

The transformation $x \mapsto x - \mu$ transforms (1) into (4). The corresponding dynamic partitions of (1) and (4) are mapped onto each other by this transformation, that is, the type of (t, x) for (1) and $\big(t, x - \mu(t)\big)$ for (4) are identical for all $t \in I$. Let $Q : I \to \mathbb{R}^{n \times n}$ be a continuously differentiable function of orthogonal matrices. We transform (4) by the transformation $x \mapsto Q^\top x$, that is, we compute (4) in the new coordinates $\widetilde{x} = Q^\top x$, for notational convenience omit the tilde, and obtain

$$\dot{x} = \big[Q(t)^\top D_x f\big(t, \mu(t)\big)Q(t) - Q(t)^\top \dot{Q}(t)\big]x + h(t, x)\,, \tag{9}$$

with $h(t, x) = Q(t)^\top \big[f\big(t, Q(t)x + \mu(t)\big) - f\big(t, \mu(t)\big) - D_x f\big(t, \mu(t)\big)Q(t)x\big]$. The same transformation $x \mapsto Q(t)^\top x$ transforms the linearization (5) of (1) into the linearization of (9) along the zero solution,

$$\dot{\xi} = B(t)\xi\,, \tag{10}$$

where $B(t) = Q(t)^\top D_x f\big(t, \mu(t)\big) Q(t) - Q(t)^\top \dot{Q}(t)$. If $Q(t)$ commutes with Γ for all $t \in I$, then the dynamic partitions corresponding to (4) and (9) are mapped onto each other by the transformation $x \mapsto Q^\top x$. In other words, the time-dependent shift and orthogonal transformation $x \mapsto Q(t)^\top [x - \mu(t)]$ transforms system (1) and its dynamic partition into system (9) and its dynamic partition, respectively. This observation is made precise by the following lemma which shows that in fact any time-dependent shift and orthogonal transformation $x \mapsto Q^\top(x - u)$ with $Q(t)\Gamma = \Gamma Q(t)$ for all $t \in I$ preserves the dynamic partition.

Lemma 2.5 (Dynamic Partition under Shift and Orthogonal Transformation). *Consider the equation*

$$\dot{x} = f(t, x), \tag{11}$$

together with a symmetric, positive definite matrix Γ, and let $Q : I \to \mathbb{R}^{n \times n}$ and $u : I \to \mathbb{R}^n$ be a C^1 function of orthogonal matrices and a C^1 function, respectively. Moreover, assume that Q commutes with Γ, i.e. $Q(t)\Gamma = \Gamma Q(t)$ for all $t \in I$. Then the transformation $x \mapsto Q(t)^\top [x - u(t)]$ transforms (11) into

$$\dot{x} = Q(t)^\top f\big(t, Q(t)x + u(t)\big) - Q(t)^\top \dot{Q}(t)x - Q(t)^\top \dot{u}(t), \tag{12}$$

and for any x solution of (11), $\widetilde{x} = Q^\top [x - u]$ is a solution of (12). Moreover, the linearization of (12) along $Q^\top [x - u]$ is given by

$$\dot{\xi} = \big[Q(t)^\top D_x f\big(t, x(t)\big) Q(t) - Q(t)^\top \dot{Q}(t)\big]\xi,$$

and, for every $t \in I$, the points $\big(t, x(t)\big)$ for (11) and $(t, Q^\top(t)[x(t) - u(t)])$ for (12) have the same type w.r.t. $\|\cdot\|_\Gamma$.

Proof. Let x be a solution of (11). Then $\widetilde{x} = Q^\top [x - u]$ is a solution of (12). Let S_Γ and M_Γ denote the Γ-strain and Γ-strain acceleration tensor of (11) along x. With $\widetilde{A}(t) = Q(t)^\top D_x f\big(t, x(t)\big) Q(t) - Q(t)^\top \dot{Q}(t)$,

$$\widetilde{S}_\Gamma(t) = \frac{1}{2}\left[\Gamma \widetilde{A}(t) + \widetilde{A}(t)^\top \Gamma\right] \quad \text{and} \quad \widetilde{M}_\Gamma(t) = \dot{\widetilde{S}}_\Gamma(t) + \widetilde{S}_\Gamma(t)\widetilde{A}(t) + \widetilde{A}(t)^\top \widetilde{S}_\Gamma(t)$$

are, respectively, the Γ-strain and Γ-strain acceleration tensor of (12) along $\widetilde{x}$. Omitting the argument t for ease of notation, note that $Q^\top \Gamma = \Gamma Q^\top$ and also $\dot{Q}\Gamma = \Gamma \dot{Q}$; furthermore, $Q^\top Q = I$ and hence $\dot{Q}^T = -Q^T \dot{Q} Q^T$. From

$$\begin{aligned}
\widetilde{S}_\Gamma &= \frac{1}{2}\left[\Gamma \widetilde{A} + \widetilde{A}^\top \Gamma\right] \\
&= \frac{1}{2}\left[\Gamma(Q^\top A Q - Q^\top \dot{Q}) + (Q^\top A^\top Q - \dot{Q}^\top Q)\Gamma\right] \\
&= \frac{1}{2}\left[Q^\top \Gamma(A - \dot{Q}Q^\top)Q + Q^\top(A^\top + \dot{Q}Q^\top)\Gamma Q\right] \\
&= \frac{1}{2}Q^\top \left[\Gamma(A - \dot{Q}Q^\top) + (A^\top + \dot{Q}Q^\top)\Gamma\right] Q \\
&= \frac{1}{2}Q^\top \left[\Gamma A + A^\top \Gamma\right] Q = Q^\top S_\Gamma Q,
\end{aligned}$$

it follows that $\widetilde{S}_\Gamma$ and S_Γ determine congruent quadratic forms, that is, $\xi \in Z_\Gamma$ if and only if $Q^\top \xi \in \widetilde{Z}_\Gamma$. A similar computation shows that $\widetilde{M}_\Gamma = Q^\top M_\Gamma Q$, and since the dynamic partition is defined exclusively in terms of the Γ-strain and Γ-strain acceleration tensors, the proof is complete. $\square$

If Lemma 2.5 is applied to the linearization (5) with $Q(t)$ consisting of an orthonormal basis of eigenvectors of $S_\Gamma\big(t, \mu(t)\big)$ then the transformation $x \mapsto Q(t)^\top x$ factors out that part of the time dependence of $D_x\big(t, \mu(t)\big)$ that comes from the rotation of the eigenvectors of the Γ-strain tensor S_Γ, provided that Q and Γ commute. In the fluid dynamics literature this transformation is, for the case $\Gamma = \mathrm{id}_{n \times n}$, called a *strain coordinate transformation* (see e.g. [3]).

Definition 2.6 (Strain Coordinates). Consider equation (11) and a solution $\mu : I \to \mathbb{R}^n$. Suppose that $Q : I \to \mathbb{R}^{n \times n}$ is a continuously differentiable function of orthogonal matrices such that $Q^\top(t) S_\Gamma\big(t, \mu(t)\big) Q(t)$ is diagonal for all $t \in I$, and also $Q(t)\Gamma = \Gamma Q(t)$. Then $x \mapsto Q^\top[x - \mu]$ is called a *strain coordinate transformation*. The transformed equations (9) and (10) are said to be *in strain coordinates*.

If (9) is in strain coordinates then the Γ-strain tensor corresponding to $B(t)$ in (10) is diagonal. In the literature the existence of a strain coordinate transformation is often assumed to simplify computations. Reasonable though it may be, this assumption requires justification, which in fact may have its subtleties. For instance, if $\Gamma = \mathrm{id}_{n \times n}$ and $t \mapsto S(t)$ is analytic in t then [15, Theorem 6.1] implies that the normalized eigenvectors of $S(t)$ are also analytic. Hence a strain coordinate transformation exists for (1) along a solution μ provided $t \mapsto A(t) = D_x f\big(t, \mu(x)\big)$, and therefore also the symmetric part $S(t)$, is analytic. On the other hand, if $t \mapsto S(t)$ is merely C^∞ then [15, Example 5.3] shows that in general the associated normalized eigenvectors cannot even be continued as continuous functions. The obstruction for the existence of strain coordinates comes in the form of multiple eigenvalues, as can be seen from the following simple example (cf. [4]). The zero solution $\mu = 0$ of $\dot{x} = A(t)x$ with

$$A(t) = \begin{pmatrix} -1 & \frac{1}{2}t^2 & 0 \\ \frac{1}{2}t^2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for } t < 0 , \quad A(t) = \begin{pmatrix} -1 + \frac{1}{2}t^2 & 0 & 0 \\ 0 & -1 + \frac{1}{2}t^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for } t \geq 0 ,$$

satisfies $\mu(t) \in \mathcal{H}(t)$ for all $t \in [-1, 1]$ w.r.t. the Euclidean norm. However, the unique (up to a permutation of columns) orthogonal matrix $Q(t)$ consisting of the normalized eigenvectors of $A(t)$ is given by

$$Q(t) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for } t < 0 \quad \text{and} \quad Q(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for } t \geq 0 ,$$

and therefore is not even continuous. As a consequence, the equation $\dot{x} = A(t)x$ cannot be transformed into strain coordinates on $[-1, 1]$. Note that an explicit strain coordinate transformation w.r.t. the Euclidean norm can be constructed if the eigenvalues of S stay separated, see [4] for an explicit formula for $n = 2$. Although strain coordinates may not exist, one can nevertheless apply Lemma 2.5 to (11) for each *fixed* $t_0 \in I$ to factor out the rotation of the zero strain set, i.e., choose $Q = \big(v_1 | \cdots | v_n\big)$ with an orthonormal basis of eigenvectors v_i of $S(t_0)$, provided that $Q(t_0)\Gamma = \Gamma Q(t_0)$. We show in the next lemma that one can always ensure $\Gamma = \mathrm{id}_{n \times n}$ by means of an appropriate transformation.

Lemma 2.7 (Dynamic Partition under Linear Transformation). *Let Γ and T be a symmetric, positive definite and an invertible matrix, respectively. Consider*

$$\dot{x} = f(t, x) \qquad \text{with norm } \| \cdot \|_\Gamma = \langle \cdot, \Gamma \cdot \rangle^{\frac{1}{2}} , \tag{13}$$

as well as the transformed equation

$$\dot{\widetilde{x}} = T^{-1} f(t, T\widetilde{x}) \qquad \text{with norm } \| \cdot \|_{\widetilde{\Gamma}} = \langle \cdot, \widetilde{\Gamma} \cdot \rangle^{\frac{1}{2}}, \tag{14}$$

where $\widetilde{\Gamma} = T^{\top} \Gamma T$. Then (t_0, x_0) for (13) and $(t_0, \widetilde{x}_0) = (t_0, T^{-1} x_0)$ for (14) have the same type, that is, $(t_0, x_0) \in \mathcal{T}_{\Gamma}$ if and only if $(t_0, \widetilde{x}_0) \in \mathcal{T}_{\widetilde{\Gamma}}$. Moreover, with $\widetilde{x}_1 = T^{-1} x_1$,

$$\frac{d}{dt} \|x(t, t_0, x_1) - x(t, t_0, x_0)\|_{\Gamma}^2 \big|_{t=t_0} = \frac{d}{dt} \|\widetilde{x}(t, t_0, \widetilde{x}_1) - \widetilde{x}(t, t_0, \widetilde{x}_0)\|_{\widetilde{\Gamma}}^2 \big|_{t=t_0},$$

where $x(t, t_0, x_0)$ and $\widetilde{x}(t, t_0, \widetilde{x}_0)$ denote, respectively, the solutions of (13) and (14) starting at t_0 in x_0 and $\widetilde{x}_0$.

Proof. Pick $(t_0, x_0) \in I \times \mathbb{R}^n$. Let $S_{\Gamma} = S_{\Gamma}(t_0, x_0)$ and $M_{\Gamma} = M_{\Gamma}(t_0, x_0)$ denote the Γ-strain and Γ-strain acceleration tensor of (13) at (t_0, x_0). With $A = D_x f(t_0, x_0)$ and $\widetilde{A} = D_{\widetilde{x}} [T^{-1} f(t, T\widetilde{x})] \big|_{(t_0, \widetilde{x}_0)} = T^{-1} A T$ the $\widetilde{\Gamma}$-strain and $\widetilde{\Gamma}$-strain acceleration tensor of (14) at $(t_0, \widetilde{x}_0)$ are

$$\widetilde{S}_{\Gamma} = \frac{1}{2} \left[\widetilde{\Gamma} \widetilde{A} + \widetilde{A}^{\top} \widetilde{\Gamma} \right] \quad \text{and} \quad \widetilde{M}_{\Gamma} = \dot{\widetilde{S}}_{\Gamma} + \widetilde{S}_{\Gamma} \widetilde{A} + \widetilde{A}^{\top} \widetilde{S}_{\Gamma},$$

respectively. From

$$\widetilde{S}_{\Gamma} = \tfrac{1}{2} [T^{\top} \Gamma T T^{-1} A T + T^{\top} A^{\top} (T^{-1})^{\top} T^{\top} \Gamma T] = T^{\top} S_{\Gamma} T, \tag{15}$$

it follows that $\widetilde{S}_{\Gamma}$ and S_{Γ} determine congruent quadratic forms, hence $\widetilde{Z}_{\Gamma}(t_0, \widetilde{x}_0) = T^{-1} Z_{\Gamma}(t_0, x_0)$. Similarly, $\widetilde{M}_{\Gamma} = T^{\top} M_{\Gamma} T$ holds, and therefore (t_0, x_0) for (13) and $(t_0, \widetilde{x}_0)$ for (14) have the same type. The proof is completed by observing that

$$\frac{d}{dt} \|x(t, t_0, x_1) - x(t, t_0, x_0)\|_{\Gamma}^2 \big|_{t=t_0} = 2 \langle x_1 - x_0, \Gamma f(t_0, x_1) - \Gamma f(t_0, x_0) \rangle$$

$$= 2 \left\langle \widetilde{x}_1 - \widetilde{x}_0, \widetilde{\Gamma} T^{-1} f(t_0, T\widetilde{x}_1) - \widetilde{\Gamma} T^{-1} f(t_0, T\widetilde{x}_0) \right\rangle$$

$$= \frac{d}{dt} \|\widetilde{x}(t, t_0, \widetilde{x}_1) - \widetilde{x}(t, t_0, \widetilde{x}_0)\|_{\widetilde{\Gamma}}^2 \big|_{t=t_0},$$

for every $\widetilde{x}_1 = T^{-1} x_1$. $\qquad \square$

In the next lemma we show that all but the degenerate part of the dynamic partition are actually open sets.

Lemma 2.8. *The sets $\mathcal{A}_{\Gamma}, \mathcal{R}_{\Gamma}, \mathcal{E}_{\Gamma}, \mathcal{H}_{\Gamma}$ and $\mathcal{Q}_{\Gamma}$ are open in $I \times \mathbb{R}^n$.*

Proof. Assume first that $(t_0, x_0) \in \mathcal{A}_{\Gamma}$ or $(t_0, x_0) \in \mathcal{R}_{\Gamma}$. Then $S_{\Gamma}(t_0, x_0)$ is negative or positive definite, and by continuity the same is true for $S_{\Gamma}(t, x)$ provided that (t, x) is sufficiently close to (t_0, x_0). Hence $\mathcal{A}_{\Gamma}$ and $\mathcal{R}_{\Gamma}$ are both open.

To deal with the remaining cases, let

$$\Sigma = \{(t, x) \in I \times \mathbb{R}^d : S_{\Gamma}(t, x) \text{ is indefinite and non-degenerate}\}.$$

Clearly, Σ is open in $I \times \mathbb{R}^d$, and $\mathcal{E}_{\Gamma} \cup \mathcal{H}_{\Gamma} \cup \mathcal{Q}_{\Gamma} \subset \Sigma$. For each $(t, x) \in \Sigma$ define the continuous function $\psi_{(t,x)} : S^{d-1} \to \mathbb{R}$ by $\psi_{(t,x)}(\xi) = \langle \xi, M_{\Gamma}(t, x)\xi \rangle$. Obviously, the map $(t, x) \mapsto \psi_{(t,x)} \in C(S^{d-1})$ is continuous; here, as usual, $C(S^{d-1})$ denotes the Banach space of all continuous real-valued functions on S^{d-1}. Also, for each $(t, x) \in \Sigma$ the set $F(t, x) = \{\xi \in S^{d-1} : \langle \xi, S_{\Gamma}(t, x)\xi \rangle = 0\} = Z_{\Gamma}(t, x) \cap S^{d-1}$ is non-empty and compact, hence $F(t, x)$ is an element of $\mathcal{K}(S^{d-1})$, the (complete metric) space of all non-empty compact subsets of S^{d-1}. Moreover, the map $(t, x) \mapsto F(t, x) \in \mathcal{K}(S^{d-1})$ is also continuous. To see this, assume by way of contradiction

that $(t_n, x_n) \to (t_0, x_0)$ yet $d_H\big(F(t_n, x_n), F(t_0, x_0)\big) \geq \delta$ for all n and some $\delta > 0$. (Here d_H denotes the Hausdorff distance in $\mathcal{K}(S^{d-1})$ as induced by the standard Euclidean metric on S^{d-1}.) In this case, there exists a sequence (ξ_n) in S^{d-1} such that $\inf_{\xi \in F(t_0, x_0)} \|\xi - \xi_n\| \geq \delta$ but also $\xi_n \in F(t_n, x_n)$, that is, $\langle \xi_n, S_\Gamma(t_n, x_n)\xi_n \rangle = 0$, for all n. Assume without loss of generality that $\xi_n \to \xi^*$. Then, by continuity, $\langle \xi^*, S_\Gamma(t_0, x_0)\xi^* \rangle = 0$ and thus $\xi^* \in F(t_0, x_0)$ yet $\inf_{\xi \in F(t_0, x_0)} \|\xi - \xi^*\| \geq \delta$, an obvious contradiction. Hence $(t, x) \mapsto F(t, x)$ is continuous on Σ. Finally, for each $g \in C(S^{d-1})$ and $K \in \mathcal{K}(S^{d-1})$ let $\Psi^+(g, K) = \max_{\xi \in K} g(\xi)$ and $\Psi^-(g, K) = \min_{\xi \in K} g(\xi)$. It is easy to see that Ψ^+ and Ψ^- constitute continuous real-valued functions on $C(S^{d-1}) \times \mathcal{K}(S^{d-1})$. From the above it follows that the function $(t, x) \mapsto \Psi^+\big(\psi_{(t,x)}, F(t, x)\big) =: \Psi^+(t, x)$ from Σ into $\mathbb{R}$ is continuous, as is the analogously defined function Ψ^-. Observing that $\mathcal{E}_\Gamma = \{(t, x) \in X : \Psi^+ \cdot \Psi^- < 0\}$, $\mathcal{H}_\Gamma = \{(t, x) \in X : \Psi^- > 0\}$, and $\mathcal{Q}_\Gamma = \{(t, x) \in X : \Psi^+ < 0\}$ therefore completes the proof. $\qquad \square$

Our next result describes the dynamic partition for rest points of autonomous equations $\dot{x} = F(x)$. If $F(x_0) = 0$ then the linearization at x_0, that is $\dot{\xi} = A\xi$ with $A = D_x F(x_0)$, is autonomous. For any given Γ the latter has, according to Remark 10, one and the same type $\mathcal{T}_\Gamma$ for all $(t, x) \in I \times \mathbb{R}^n$. The following theorem characterizes this type in terms of the eigenvalues of A. To formulate the statement concisely, let $\sigma(A)$ denote the spectrum of A, i.e. the set of all eigenvalues of A, and let $\mathbb{C}^- = \{z \in \mathbb{C} : \Re z < 0\}$, $\mathbb{C}^+ = -\mathbb{C}^-$, and $i\mathbb{R}$ symbolize, respectively, the open left half-plane, the open right half-plane, and the imaginary axis.

Theorem 2.9 (Dynamic Partition for Linear Systems with Constant Coefficients). *Let $A \in \mathbb{R}^{n \times n}$. For the linear equation*

$$\dot{x} = Ax \tag{16}$$

the following holds:

   (i) $\mathcal{Q}_\Gamma = \emptyset$ for all Γ, i.e., (16) is never quasihyperbolic.
   (ii) $\sigma(A) \subset \mathbb{C}^-$ if and only if $\mathcal{A}_\Gamma = I \times \mathbb{R}^n$ (i.e., (16) is attracting) for some Γ.
   (iii) $\sigma(A) \subset \mathbb{C}^+$ if and only if $\mathcal{R}_\Gamma = I \times \mathbb{R}^n$ (i.e., (16) is repelling) for some Γ.
   (iv) If $\sigma(A) \subset \mathbb{C}^-$ or $\sigma(A) \subset \mathbb{C}^+$ then $\mathcal{E}_\Gamma = I \times \mathbb{R}^n$ for some Γ, unless $A = \alpha \, \mathrm{id}_{n \times n}$ with $\alpha \in \mathbb{R} \backslash \{0\}$.
   (v) $\sigma(A) \cap \mathbb{C}^-$ and $\sigma(A) \cap \mathbb{C}^+$ are both non-empty while $\sigma(A) \cap i\mathbb{R} = \emptyset$ (that is, A has eigenvalues on both sides of the imaginary axis but none on it) if and only if $\mathcal{H}_\Gamma = I \times \mathbb{R}^n$ for some Γ.
   (vi) $\sigma(A) \cap i\mathbb{R} \neq \emptyset$ if and only if for every Γ either $\mathcal{E}_\Gamma = I \times \mathbb{R}^n$ or $\mathcal{D}_\Gamma = I \times \mathbb{R}^n$, i.e., (16) is always either elliptic or degenerate.

Proof. (i) Assume that $\mathcal{Q}_\Gamma = I \times \mathbb{R}^n$. Then $n \geq 2$, and according to Lemma 2.5 and 2.7 it can be assumed that $\Gamma = \mathrm{id}_{n \times n}$ and S is diagonal, $S = \mathrm{diag}(\mu_1, \ldots, \mu_n)$ with $\mu_i \neq 0$ for all i and $\mu_1 > 0 > \mu_n$. From $S = \frac{1}{2}(A + A^\top)$ it follows that

$$a_{ii} = \mu_i \quad \text{and} \quad a_{ij} + a_{ji} = 0 \quad \text{for all } i \neq j \,.$$

Furthermore, $M = SA + A^\top S$, so that

$$\tfrac{1}{2}\langle \xi, M\xi \rangle = \sum_{i=1}^n \mu_i^2 \xi_i^2 + \sum_{i<j}(\mu_i a_{ij} + \mu_j a_{ji})\xi_i \xi_j \,.$$

With $\xi_\pm = (1/\sqrt{\mu_1}, 0, \ldots, 0, \pm 1/\sqrt{-\mu_n})^\top$ clearly $\xi_\pm \in Z\backslash\{0\}$, and

$$\tfrac{1}{2}\langle \xi_\pm, M\xi_\pm \rangle = \mu_1 - \mu_n \pm (\mu_1 - \mu_n)\frac{a_{1n}}{\sqrt{-\mu_1 \mu_n}}\,.$$

Thus by letting ξ equal ξ_+ or ξ_-, depending on whether $a_{1n} \geq 0$ or $a_{1n} < 0$, we can find a point $\xi \in Z\backslash\{0\}$ with $\langle \xi, M\xi \rangle > 0$. Since this contradicts the assumed quasihyperbolicity, $\mathcal{Q}_\Gamma = \emptyset$.

(ii) Assume first that $\sigma(A) \subset \mathbb{C}^-$, that is, all eigenvalues of A have negative real part. To find $\Gamma = \Gamma^\top > 0$ such that $\mathcal{A}_\Gamma = I \times \mathbb{R}^n$, choose a regular real matrix P such that $P^{-1}AP$ has real Jordan normal form. Since the subsequent argument can be applied independently to each block in the normal form, no generality is lost in assuming that either

$$P^{-1}AP = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix} \in \mathbb{R}^{n \times n},$$

with some $\lambda < 0$, or that $n = 2m$, and

$$P^{-1}AP = \begin{pmatrix} \Re\lambda & 1 & & & \Im\lambda & & & \\ & \ddots & \ddots & & & \ddots & & \\ & & \Re\lambda & 1 & & & \Im\lambda & \\ & & & \Re\lambda & & & & \Im\lambda \\ -\Im\lambda & & & & \Re\lambda & 1 & & \\ & \ddots & & & & \ddots & \ddots & \\ & & -\Im\lambda & & & & \Re\lambda & 1 \\ & & & -\Im\lambda & & & & \Re\lambda \end{pmatrix} \in \mathbb{R}^{2m \times 2m},$$

with $\Re\lambda < 0$ and $\Im\lambda > 0$. In the first case let $P_\varepsilon = \mathrm{diag}(1, \varepsilon, \ldots, \varepsilon^{n-1})$ for any $\varepsilon > 0$, whereas in the second case let

$$P_\varepsilon = \mathrm{diag}(1, \varepsilon, \ldots, \varepsilon^{m-1}, 1, \varepsilon, \ldots, \varepsilon^{m-1})\,. \tag{17}$$

In either case, define $B = P_\varepsilon^{-1}P^{-1}APP_\varepsilon$ and deduce from a short computation that

$$\langle \widetilde{x}, \tfrac{1}{2}(B + B^\top)\widetilde{x} \rangle \leq (\Re\lambda + \varepsilon)\|\widetilde{x}\|^2 = -(|\Re\lambda| - \varepsilon)\|\widetilde{x}\|^2 \qquad \text{for all } \widetilde{x} \in \mathbb{R}^n\,.$$

If we, therefore, choose $0 < \varepsilon < |\Re\lambda|$ then $\dot{\widetilde{x}} = B\widetilde{x}$ is attracting with respect to the Euclidean norm. By Lemma 2.7, (16) is attracting with respect to $\|\cdot\|_\Gamma$ where $\Gamma = (P^{-1})^\top(P_\varepsilon^{-1})^\top P_\varepsilon^{-1}P^{-1}$.

Conversely, assume that $\mathcal{A}_\Gamma = I \times \mathbb{R}^n$. Given a *real* eigenvalue λ of A, let v be any (non-zero) eigenvector corresponding to λ and observe that

$$0 > \langle v, S_\Gamma v \rangle = \langle v, \Gamma A \rangle = \lambda \langle v, \Gamma v \rangle\,.$$

Since Γ is positive definite, $\lambda < 0$. Given a *non-real* eigenvalue λ of A, there exist two linearly independent vectors v_1, v_2 such that

$$Av_1 = \Re\lambda\, v_1 - \Im\lambda\, v_2\,, \quad Av_2 = \Im\lambda\, v_1 + \Re\lambda\, v_2\,.$$

It follows from

$$0 > \langle v_1, S_\Gamma v_1 \rangle = \Re\lambda\, \langle v_1, \Gamma v_1 \rangle - \Im\lambda\, \langle v_1, \Gamma v_2 \rangle\,,$$

that $\Im\lambda\langle v_1, \Gamma v_2\rangle > \Re\lambda\langle v_1, \Gamma v_1\rangle$ and therefore

$$0 > \langle v_2, S_\Gamma v_2\rangle = \Im\lambda\langle v_1, \Gamma v_2\rangle + \Re\lambda\langle v_2, \Gamma v_2\rangle > \Re\lambda(\langle v_1, \Gamma v_1\rangle + \langle v_2, \Gamma v_2\rangle),$$

hence $\Re\lambda < 0$. Thus $\Re\lambda < 0$ for every eigenvalue of A, that is, $\sigma(A) \subset \mathbb{C}^-$.

(iii) Since $\sigma(A) \subset \mathbb{C}^+$ if and only if $\sigma(-A) \subset \mathbb{C}^-$, the statement follows immediately from (ii) applied to $-A$.

(iv) If $A = \alpha\,\mathrm{id}_{n\times n}$ with some $\alpha \neq 0$ then (16) is attracting or repelling for every Γ, depending on whether $\alpha < 0$ or $\alpha > 0$. In any other case, however, we are going to show that (16) is elliptic for some Γ. Three cases will be studied: Either A has two different real eigenvalues of the same sign each of which corresponds to a trivial (i.e. 1×1) Jordan block, or A has a pair of complex-conjugate eigenvalues with non-zero real part, or A has an eigenvalue off the imaginary axis corresponding to a non-trivial Jordan block, i.e., with geometric multiplicity less than algebraic multiplicity. In general, if A is not a multiple of $\mathrm{id}_{n\times n}$ and satisfies $\sigma(A) \subset \mathbb{C}^-$ or $\sigma(A) \subset \mathbb{C}^+$, then at least one of these cases occurs. Thus the proof will be complete once we demonstrate for every case how to find Γ such that (16) is elliptic with respect to $\|\cdot\|_\Gamma$. Each case will be dealt with separately.

Assume first that A has two different real eigenvalues $\lambda_1 > \lambda_2$ with $\lambda_1\lambda_2 > 0$, each corresponding to a trivial Jordan block. According to Lemma 2.7 no generality is lost in assuming that $Ae_1 = \lambda_1 e_1$, $Ae_2 = \lambda_2 e_2$, where e_i denotes the i-th element of the canonical basis of $\mathbb{R}^n$, and Ae_i is contained in the linear span of $e_3, \ldots, e_n$ whenever $i \geq 3$. To ensure that $x \mapsto \langle x, Ax\rangle$ is definite on that linear span, we can proceed as in (ii) and use matrices of the type P_ε which, however, must not affect the span of e_1 and e_2. Define $\Gamma e_1 = e_1 + \alpha e_2$, $\Gamma e_2 = \alpha e_1 + \beta e_2$, where the numbers $\alpha, \beta > 0$ have yet to be determined, and $\Gamma e_i = e_i$ for all $i \geq 3$. It is readily confirmed that $\Gamma = \Gamma^\top$ is positive definite provided that $\beta/\alpha^2 > 1$. On the other hand, S_Γ is indefinite and non-degenerate whenever $\beta/\alpha^2 < 1 + (\lambda_1 - \lambda_2)^2/(4\lambda_1\lambda_2)$. Thus choosing for instance $\alpha = \sqrt{2}$ and $\beta = 2 + (\lambda_1 - \lambda_2)^2/(4\lambda_1\lambda_2)$ we obtain, for all s, t,

$$\langle se_1 + te_2, S_\Gamma(se_1 + te_2)\rangle = \lambda_1 s^2 + \sqrt{2}(\lambda_1 + \lambda_2)st + \frac{\lambda_1^2 + 6\lambda_1\lambda_2 + \lambda_2^2}{4\lambda_1}t^2\,,$$

so that $se_1 + te_2 \in Z_\Gamma\backslash\{0\}$ provided that $s\lambda_1/t = -\frac{1}{\sqrt{2}}(\lambda_1 + \lambda_2) \pm \frac{1}{2}|\lambda_1 - \lambda_2|$ and $st \neq 0$. Moreover, for this particular choice of s, t we find

$$\langle se_1 + te_2, M_\Gamma(se_1 + te_2)\rangle = \pm st(\lambda_1 - \lambda_2)^2 \neq 0\,,$$

which shows that (16) is elliptic with respect to $\|\cdot\|_\Gamma$.

Next assume that A has a pair of complex-conjugate eigenvalues $\lambda, \overline{\lambda}$ with non-vanishing real part. Invoking Lemma 2.7 again, and replacing A by $-A$ if necessary, we may assume that $Ae_1 = \Re\lambda\,e_1 - \Im\lambda\,e_2$, $Ae_2 = \Im\lambda\,e_1 + \Re\lambda\,e_2$ with $\Re\lambda, \Im\lambda > 0$, and Ae_i is contained in the linear span of $e_3, \ldots, e_n$ for all $i \geq 3$. As before, we can ensure that $\langle\cdot, A\cdot\rangle$ is definite on the latter space, and choosing Γ as above, with α, β such that

$$\frac{\Re\lambda}{\Im\lambda} < \alpha < \frac{\Re\lambda + |\lambda|}{\Im\lambda} \quad\text{and}\quad \beta = 1 + 2\alpha\frac{\Re\lambda}{\Im\lambda}\,,$$

we deduce from a completely analogous computation that M_{Z_Γ} is indefinite, and hence (16) is elliptic with respect to $\|\cdot\|_\Gamma$.

Finally, if A has an eigenvalue λ for which algebraic and geometric multiplicity do not coincide, then we may distinguish two subcases. If $\Re\lambda, \Im\lambda > 0$ then $n = 2m$

and it can be assumed that

$$Ae_1 = \Re\lambda\, e_1 - \Im\lambda\, e_{m+1}, \qquad\qquad Ae_2 = e_1 + \Re\lambda\, e_1 - \Im\lambda\, e_{m+2},$$
$$Ae_{m+1} = \Im\lambda\, e_1 + \Re\lambda\, e_{m+1}, \qquad\qquad Ae_{m+2} = \Im\lambda\, e_2 + e_{m+1} + \Re\lambda\, e_{m+2},$$

that is, A is in real Jordan normal form. With $\Gamma e_1 = e_1 + \alpha e_{m+1}$, $\Gamma e_{m+1} = \alpha e_1 + \beta e_{m+1}$, and $\Gamma e_i = e_i$ for all $i \notin \{1, m+1\}$ the same argument as in the second case above shows that M_{Z_Γ} is indefinite. On the other hand, if $\lambda \in \mathbb{R}\backslash\{0\}$ then we can assume that $Ae_1 = \lambda e_1$, $Ae_2 = e_1 + \lambda e_2$. In this case, letting $\Gamma e_1 = \alpha e_1$ with $\alpha > 0$, and $\Gamma e_i = e_i$ for all $i \geq 2$, yields a symmetric positive definite matrix Γ for which S_Γ is indefinite whenever $\alpha > 4\lambda^2$, and M_{Z_Γ} attains positive and negative values. (In either case it may again be necessary to proceed as above to ensure that S_Γ is non-degenerate.)

(v) Assume first that $\sigma(A) \cap \mathbb{C}^-$ and $\sigma(A) \cap \mathbb{C}^+$ are both non-empty while $\sigma(A) \cap i\mathbb{R} = \emptyset$. As in (ii), choose P such that $P^{-1}AP$ has real Jordan normal form. Again it is enough to deal with a single Jordan block. Note also that

$$\langle x, M_\Gamma x\rangle = \langle x, (\Gamma A^2 + A^\top \Gamma A)x\rangle = \langle x, \Gamma A^2 x\rangle + \langle Ax, \Gamma Ax\rangle \geq \langle x, \Gamma A^2 x\rangle,$$

so that the proof can be shortened if $\langle x, \Gamma A^2 x\rangle > 0$ holds for some $\Gamma = \Gamma^\top > 0$ and all $x \neq 0$. Assume first that

$$P^{-1}AP = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix} \in \mathbb{R}^{n \times n},$$

with some $\lambda \in \mathbb{R}\backslash\{0\}$. With $P_\varepsilon = \mathrm{diag}(1, \varepsilon, \ldots, \varepsilon^{n-1})$ and $B = P_\varepsilon^{-1} P^{-1} A P P_\varepsilon$ it is straightforward to check that

$$\langle \widetilde{x}, B^2 \widetilde{x}\rangle \geq (\lambda^2 - 2\varepsilon|\lambda| - \varepsilon^2)\|\widetilde{x}\|^2 \qquad \text{for all } \widetilde{x} \in \mathbb{R}^n,$$

so that $\widetilde{x} \mapsto \langle \widetilde{x}, B^2 \widetilde{x}\rangle$ is positive definite provided that $0 < \varepsilon < (\sqrt{2}-1)|\lambda|$. Hence (16) is hyperbolic for all sufficiently small $\varepsilon > 0$ and Γ as in (ii). Similarly, if

$$P^{-1}AP = \begin{pmatrix} \Re\lambda & 1 & & & \Im\lambda & & & \\ & \ddots & \ddots & & & \ddots & & \\ & & \Re\lambda & 1 & & & \Im\lambda & \\ & & & \Re\lambda & & & & \Im\lambda \\ -\Im\lambda & & & & \Re\lambda & 1 & & \\ & \ddots & & & & \ddots & \ddots & \\ & & -\Im\lambda & & & & \Re\lambda & 1 \\ & & & -\Im\lambda & & & & \Re\lambda \end{pmatrix} \in \mathbb{R}^{2m \times 2m},$$

with $n = 2m$ and $\Re\lambda, \Im\lambda \in \mathbb{R}\backslash\{0\}$, then choosing P_ε according to (17) and letting $B = P_\varepsilon^{-1} P^{-1} A P P_\varepsilon$ again yields

$$\langle \widetilde{x}, (B^2 + B^\top B)\widetilde{x}\rangle \geq \big((\Re\lambda)^2 - (2|\Re\lambda| + |\Im\lambda|)\varepsilon - \varepsilon^2\big)\|\widetilde{x}\|^2 \qquad \text{for all } \widetilde{x} \in \mathbb{R}^n.$$

Carrying out the above for every eigenvalue of A separately, we obtain, for all sufficiently small $\varepsilon > 0$, a matrix B such that $\frac{1}{2}(B + B^\top)$ is indefinite, and $\langle \widetilde{x}, (B^2 + B^\top B)\widetilde{x}\rangle > 0$ whenever $\widetilde{x} \neq 0$. Thus $\dot{\widetilde{x}} = B\widetilde{x}$ is hyperbolic with respect to the Euclidean norm, and (16) is hyperbolic with respect to $\|\cdot\|_\Gamma$ where $\Gamma = (P^{-1})^\top (P_\varepsilon^{-1})^\top P_\varepsilon^{-1} P^{-1}$.

Conversely, assume that (16) is hyperbolic. Pick any $\eta \in \{\xi \in \mathbb{R}^n : \langle \xi, S_\Gamma \xi \rangle > 0\}$. According to Remark 3, $t \mapsto \|e^{At}\eta\|_\Gamma$ is increasing. On the other hand, if $\sigma(A) \subset \mathbb{C}^-$ then $\|e^{At}\eta\|_\Gamma \to 0$ as $t \to \infty$. This obvious contradiction shows that $\sigma(A) \not\subset \mathbb{C}^-$. Similarly, $\sigma(A) \not\subset \mathbb{C}^+$. Thus the proof of (v) will be complete once it is demonstrated that $\sigma(A) \cap i\mathbb{R}$ is empty. To this end, assume by way of contradiction that $\sigma(A) \cap i\mathbb{R} \neq \emptyset$, that is, A has eigenvalues on the imaginary axis. If $0 \in \sigma(A)$ then $Av = 0$ for some non-zero vector v, so that $\langle v, S_\Gamma v \rangle = \langle v, \Gamma Av \rangle = 0$, showing that $v \in Z_\Gamma \backslash \{0\}$ and hence $\mathcal{A}_\Gamma = \mathcal{R}_\Gamma = \emptyset$, but also

$$\langle v, M_\Gamma v \rangle = \langle v, (\Gamma A^2 + A^\top \Gamma A)v \rangle = 0,$$

so that $\mathcal{H}_\Gamma = \mathcal{Q}_\Gamma = \emptyset$. Thus either $\mathcal{E}_\Gamma$ or $\mathcal{D}_\Gamma$ equals $I \times \mathbb{R}^n$. If, on the other hand, $\lambda = ib \in \sigma(A)$ with some $b > 0$ then there exist two linearly independent vectors v_1, v_2 such that $Av_1 = -bv_2$, $Av_2 = bv_1$, and hence

$$\langle v_1, S_\Gamma v_1 \rangle = \langle v_1, \Gamma Av_1 \rangle = -b\langle v_1, \Gamma v_2 \rangle = -\langle v_2, S_\Gamma v_2 \rangle.$$

From

$$\langle \sin\varphi\, v_1 + \cos\varphi\, v_2, S_\Gamma(\sin\varphi\, v_1 + \cos\varphi\, v_2) \rangle = b\sin 2\varphi \langle v_1, \Gamma v_1 \rangle + b\cos 2\varphi \langle v_1, \Gamma v_2 \rangle,$$

it follows that $\sin\varphi\, v_1 + \cos\varphi\, v_2 \in Z_\Gamma \backslash \{0\}$ whenever

$$\tan 2\varphi = -\frac{\langle v_1, \Gamma v_2 \rangle}{\langle v_1, \Gamma v_1 \rangle}. \tag{18}$$

There exists a unique solution φ^* of (18) with $|\varphi^*| < \frac{\pi}{4}$, and $\varphi^* + \frac{\pi}{2}$ also solves (18). Consequently,

$$\langle \sin\varphi^* v_1 + \cos\varphi^* v_2, M_\Gamma(\sin\varphi^* v_1 + \cos\varphi^* v_2) \rangle =$$

$$= 2b^2 \cos 2\varphi^* \frac{\langle v_1, \Gamma v_1 \rangle^2 + \langle v_1, \Gamma v_2 \rangle^2}{\langle v_1, \Gamma v_1 \rangle} \neq 0, \tag{19}$$

and replacing φ^* by $\varphi^* + \frac{\pi}{2}$ results in multiplying (19) by -1. Therefore M_{Z_Γ} is indefinite, and (16) is either elliptic or degenerate. Since (16) was assumed to be hyperbolic, this shows that $\sigma(A) \subset \mathbb{C}^- \cup \mathbb{C}^+$ with $\sigma(A) \cap \mathbb{C}^- \neq \emptyset$ and $\sigma(A) \cap \mathbb{C}^+ \neq \emptyset$, and hence completes the proof of (v).

(vi) Assume that $\sigma(A) \cap i\mathbb{R} \neq \emptyset$. The proof of (v) above has shown that $\mathcal{E}_\Gamma = I \times \mathbb{R}^n$ or $\mathcal{D}_\Gamma = I \times \mathbb{R}^n$ in this case. Conversely, if $\sigma(A) \cap i\mathbb{R}$ is empty then exactly one of the cases (ii), (iii), and (v) applies and shows that (16) is, for an appropriate Γ, attracting, repelling, and hyperbolic, respectively, and for this particular Γ clearly $\mathcal{E}_\Gamma = \mathcal{D}_\Gamma = \emptyset$. ∎

Remark 8. The converse of (iv) does not hold, as can be seen for instance from

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix},$$

for which $\mathcal{E} = I \times \mathbb{R}^3$ yet $\sigma(A) \cap \mathbb{C}^- = \{-1\}$ and $\sigma(A) \cap \mathbb{C}^+ = \{1\}$ are both non-empty.

In the following Corollary to Theorem 2.9 we point out which cases can occur in which spectral situation and provide examples showing that all cases do actually occur.

Corollary 1. *Let* $A \in \mathbb{R}^{n \times n}$. *For the linear equation* (16) *the following statements hold:*

(i) If $\sigma(A) \subset \mathbb{C}^-$ then $\mathcal{A}_\Gamma = I \times \mathbb{R}^n$ or $\mathcal{E}_\Gamma = I \times \mathbb{R}^n$ or $\mathcal{D}_\Gamma = I \times \mathbb{R}^n$, i.e., (16) is either attracting or elliptic or degenerate.

(ii) If $\sigma(A) \subset \mathbb{C}^+$ then $\mathcal{R}_\Gamma = I \times \mathbb{R}^n$ or $\mathcal{E}_\Gamma = I \times \mathbb{R}^n$ or $\mathcal{D}_\Gamma = I \times \mathbb{R}^n$, i.e., (16) is either repelling or elliptic or degenerate.

(iii) If $\sigma(A) \cap \mathbb{C}^-$ and $\sigma(A) \cap \mathbb{C}^+$ are both non-empty while $\sigma(A) \cap i\mathbb{R} = \emptyset$ (that is, A has eigenvalues on both sides of the imaginary axis but none on it) then $\mathcal{H}_\Gamma = I \times \mathbb{R}^n$ or $\mathcal{E}_\Gamma = I \times \mathbb{R}^n$ or $\mathcal{D}_\Gamma = I \times \mathbb{R}^n$, i.e, (16) is either hyperbolic or elliptic or degenerate.

(iv) If $\sigma(A) \cap i\mathbb{R} \neq \emptyset$ then $\mathcal{E}_\Gamma = I \times \mathbb{R}^n$ or $\mathcal{D}_\Gamma = I \times \mathbb{R}^n$, i.e, (16) is either elliptic or degenerate.

Proof. The possibilities $\mathcal{R}_\Gamma = I \times \mathbb{R}^n$ and $\mathcal{H}_\Gamma = I \times \mathbb{R}^n$ in (i) are ruled out, respectively, by parts (iii) and (v) of Theorem 2.9. Similarly, the possibilities $\mathcal{A}_\Gamma = I \times \mathbb{R}^n$ and $\mathcal{H}_\Gamma = I \times \mathbb{R}^n$ in (ii) are ruled out by Theorem 2.9(ii,v). In (iii), $\mathcal{A}_\Gamma = \mathcal{R}_\Gamma = \emptyset$ by Theorem 2.9(ii,iii), and (iv) is obvious from Theorem 2.9(vi). It remains to show that all other cases can actually occur. To this end consider the following examples.

(i) Let $\Gamma = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$. For $A = \text{diag}(-1,-1)$, $S_\Gamma = -\Gamma$, and (16) is attracting. Choosing $A = \text{diag}(-1,-5)$ yields $S_\Gamma = \begin{pmatrix} -1 & -6 \\ -6 & -25 \end{pmatrix}$ and $M_\Gamma = \begin{pmatrix} 2 & 36 \\ 36 & 250 \end{pmatrix}$. Since $\xi_{1,2} = \begin{pmatrix} -6 \pm \sqrt{11} \\ 1 \end{pmatrix}$ are both contained in Z_Γ, yet $\langle \xi_1, M_\Gamma \xi_1 \rangle > 0 > \langle \xi_2, M_\Gamma \xi_2 \rangle$, the system (16) is elliptic. Finally, for $A = \text{diag}(-1, -\frac{3+\sqrt{5}}{2})$ one finds $\det S_\Gamma = 0$, and so (16) is degenerate.

(ii) Choosing Γ as in (i) and replacing A by $-A$ yields examples of (16) with $\sigma(A) \subset \mathbb{C}^+$ which are, respectively, repelling, elliptic, and degenerate.

(iii) Let $\Gamma = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. For $A = \text{diag}(-1,-1,1)$, $\det S_\Gamma > 0$ and $M_\Gamma = 2\Gamma$, so that (16) is hyperbolic. Choosing $A = \text{diag}(-1,-5,1)$ leads to $S_\Gamma = \begin{pmatrix} -1 & -6 & 0 \\ -6 & -25 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $M_\Gamma = \begin{pmatrix} 2 & 36 & 0 \\ 36 & 250 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Since $\xi_{1,2} = \begin{pmatrix} -6 \pm \sqrt{11} \\ 1 \\ 0 \end{pmatrix}$ are both contained in Z_Γ, yet $\langle \xi_1, M_\Gamma \xi_1 \rangle > 0 > \langle \xi_2, M_\Gamma \xi_2 \rangle$, the system (16) is elliptic. Finally, for $A = \text{diag}(-1, -\frac{3+\sqrt{5}}{2}, 1)$ one finds $\det S_\Gamma = 0$, and (16) is degenerate.

(iv) Choose Γ as in (iii), and let $A = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 3 & 0 \end{pmatrix}$ with $a \in \mathbb{R}$. From $\det S_\Gamma = a$ and Theorem 2.9(vi) it follows that (16) is degenerate if $a = 0$, and elliptic otherwise. (Note that the elliptic case can occur only if $n \geq 3$, see [4].) $\square$

We next consider an autonomous equation

$$\dot{x} = F(x), \qquad (20)$$

with a C^2 function $F : \mathbb{R}^n \to \mathbb{R}^n$. According to Remark 5, each fiber $\mathcal{T}_\Gamma(t)$ in the dynamic partition of (20) is independent of t. The following theorem states that a periodic orbit either lies entirely in the elliptic region or else contains a degenerate point. Note that a planar version of this result is already contained in [4].

Theorem 2.10 (Location of Periodic Orbits). *Assume that* (20) *has a nontrivial periodic solution μ with period $T > 0$. With $\mathcal{O} = \{\mu(t) : t \in [0, T]\}$ exactly one of the following two alternatives holds:*

(i) $\mathcal{O} \subset \mathcal{E}_\Gamma(0)$, i.e., the periodic orbit is contained in the elliptic fiber, or

(ii) $\mathcal{O} \cap \mathcal{D}_\Gamma(0) \neq \emptyset$, i.e., the periodic orbit intersects the degenerate fiber.

Proof. Assume that $\mathcal{O} \cap \mathcal{D}_\Gamma(0) = \emptyset$. Then the closed set $\mathcal{O}$ is contained entirely in one of the open regions $\mathcal{A}_\Gamma(0), \mathcal{R}_\Gamma(0), \mathcal{E}_\Gamma(0), \mathcal{H}_\Gamma(0)$ or $\mathcal{Q}_\Gamma(0)$. Since $\ddot{\mu}(t) = DF\big(\mu(t)\big)\dot{\mu}(t)$, the function $\dot{\mu}$ is a nontrivial T-periodic solution of the periodic linear equation

$$\dot{\xi} = A(t)\xi \qquad \text{where } A(t) = DF\big(\mu(t)\big), \tag{21}$$

with symmetric part $S_\Gamma(t)$ and zero strain set $Z_\Gamma(t)$. Assume that $\mathcal{O} \subset \mathcal{A}_\Gamma(0)$. Then $\mu(t) \in \mathcal{A}_\Gamma(0)$ for all $t \in [0, T]$ and by (6) we get the contradiction $\|\dot{\mu}(T)\|_\Gamma < \|\dot{\mu}(0)\|_\Gamma$. Similarly, $\mathcal{O} \subset \mathcal{R}_\Gamma(0)$ leads to a contradiction. Next assume that $\mathcal{O} \subset \mathcal{H}_\Gamma(0)$. Define the two cones $\Psi^-(t) := \{\xi \in \mathbb{R}^n : \langle \xi, S_\Gamma(t)\xi \rangle < 0\}$ and $\Psi^+(t) := \{\xi \in \mathbb{R}^2 : \langle \xi, S_\Gamma(t)\xi \rangle > 0\}$. If $\dot{\mu}(t) \in \Psi^-(t)$ for all $t \in [0, T]$ then (6) implies the contradiction $\|\dot{\mu}(T)\|_\Gamma < \|\dot{\mu}(0)\|_\Gamma$. Analogously, $\dot{\mu}(t) \in \Psi^+(t)$ cannot possibly hold for all $t \in [0, T]$. Consequently, there exists $t_0 \in [0, T]$ with $\dot{\mu}(t_0) \in Z_\Gamma(t_0)$. But then, by Remark 3, $\dot{\mu}(t) \in \Psi^+(t)$ for all $t > t_0$, which in turn yields the contradiction $\dot{\mu}(t_0) = \dot{\mu}(t_0 + T) \in \Psi^+(t_0 + T) = \Psi^+(t_0)$. Similarly, assuming that $\mathcal{O} \subset \mathcal{Q}_\Gamma(0)$ yields a contradiction. Thus $\mathcal{O} \subset \mathcal{E}_\Gamma(0)$ whenever $\mathcal{O} \cap \mathcal{D}(0) = \emptyset$. □

2.2. **Local results.** In this section we prove that the local behaviour of solutions of (1) can be approximated by means of the zero strain set and the dynamic partition. We fix a point $(t_0, x_0) \in I \times \mathbb{R}^n$ and denote by μ the unique solution of (1) satisfying $\mu(t_0) = x_0$. Also, let $x(t, t_0, x_0 + \bar{x})$ denote the solution of (1) with $x(t_0, t_0, x_0 + \bar{x}) = x_0 + \bar{x}$.

Theorem 2.11 (Local Attraction and Repulsion). *Consider $(t_0, x_0) \in I \times \mathbb{R}^n$ and the solution μ of (1) with $\mu(t_0) = x_0$.*
(i) If $(t_0, x_0) \in \mathcal{R}_\Gamma$ then there exists $\delta > 0$ such that

$$\frac{d}{dt}\|x(t, t_0, x_0 + \bar{x}) - \mu(t)\|_\Gamma^2\big|_{t=t_0} > 0 \qquad \text{for all } 0 < \|\bar{x}\|_\Gamma < \delta.$$

(ii) If $(t_0, x_0) \in \mathcal{A}_\Gamma$ then there exists $\delta > 0$ such that

$$\frac{d}{dt}\|x(t, t_0, x_0 + \bar{x}) - \mu(t)\|_\Gamma^2\big|_{t=t_0} < 0 \qquad \text{for all } 0 < \|\bar{x}\|_\Gamma < \delta.$$

Proof. Using Lemma 2.7 we can transform (1) together with the norm induced by Γ to (14) together with the Euclidean norm. For notational convenience we denote the latter equation again by $\dot{x} = f(t, x)$. To verify (i) assume that $(t_0, x_0) \in \mathcal{R}$. Definition 2.4 implies that there exists $\alpha > 0$ such that

$$\langle \xi, S(t_0, x_0)\xi \rangle = \langle \xi, D_x f(t_0, x_0)\xi \rangle > \alpha \qquad \text{for all } \|\xi\| = 1, \tag{22}$$

which in turn shows that

$$h(\bar{x}) := f(t_0, x_0 + \bar{x}) - f(t_0, x_0) - D_x f(t_0, x_0)\bar{x} = o(\|\bar{x}\|) \quad \text{as } \|\bar{x}\| \to 0.$$

Choose $\delta > 0$ so small that

$$|\langle \bar{x}, h(\bar{x}) \rangle| < \tfrac{1}{2}\alpha\|\bar{x}\|^2 \qquad \text{for all } \|\bar{x}\| \leq \delta. \tag{23}$$

Using (22) and (23) we deduce that for all $0 < \|\bar{x}\| < \delta$

$$
\begin{aligned}
\frac{d}{dt}\|x(t, t_0, x_0 + \bar{x}) - \mu(t)\|^2\big|_{t=t_0} &= 2\langle \bar{x}, D_x f(t_0, x_0)\bar{x} + h(\bar{x})\rangle \\
&= 2\langle \bar{x}, D_x f(t_0, x_0)\bar{x}\rangle + 2\langle \bar{x}, h(\bar{x})\rangle \\
&> \alpha\|\bar{x}\|^2 > 0.
\end{aligned}
$$

This completes the proof of (i). The argument for (ii) is completely analogous. $\square$

Next we compare the zero strain set $Z(t_0, x_0)$ which, by its definition, only depends on the linearization (5) with its nonlinear analogue, the *nonlinear zero strain set* of (1)

$$
Z_\Gamma^{\mathrm{nl}}(t_0, x_0) := \left\{ \bar{x} \in \mathbb{R}^n \ : \ \frac{d}{dt}\|x(t, t_0, x_0 + \bar{x}) - \mu(t)\|_\Gamma^2\big|_{t=t_0} = 0 \right\}.
$$

Lemma 2.12 (Relation between Nonlinear and Linear Zero Strain Set). *Consider the linearized equation* (5) *along the solution* $\mu : I \to \mathbb{R}^n$ *of* (1) *with* $\mu(t_0) = x_0$. *Suppose that* $S_\Gamma(t_0, x_0)$ *is non-degenerate and indefinite. Then, for each* $\xi \in Z_\Gamma(t_0, x_0) \setminus \{0\}$, *there exists* $\varepsilon > 0$ *and a* C^1 *curve* $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ *satisfying* $\gamma(0) = 0$, $\dot{\gamma}(0) = \xi$, *and* $\mathrm{graph}\gamma \subset Z_\Gamma^{\mathrm{nl}}(t_0, x_0)$.

Proof. As in the proof of Theorem 2.11 no generality is lost in considering $\dot{x} = f(t, x)$ together with the Euclidean norm. Applying Lemma 2.5 with $u = 0$ and an orthogonal matrix Q such that $Q^\top[D_x f(t_0, x_0) + D_x f(t_0, x_0)^\top]Q$ is diagonal, we can furthermore assume that $S(t_0, x_0)$ is diagonal,

$$
S(t_0, x_0) = \mathrm{diag}(\lambda_1, \ldots, \lambda_n), \tag{24}
$$

with the eigenvalues λ_i ordered according to (7) with $n^+ \in \{1, \ldots, n-1\}$. Let $\xi = (\xi_1, \ldots, \xi_n) \in Z(t_0, x_0) \setminus \{0\}$. Then

$$
\lambda_1 \xi_1^2 + \lambda_2 \xi_2^2 + \ldots + \lambda_n \xi_n^2 = 0, \tag{25}
$$

and there exists at least one $i \in \{1, \ldots, n^+\}$ with $\xi_i \neq 0$. Assume w.l.o.g. that $\xi_1 \neq 0$. The function $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ will be of the form

$$
\gamma(s) = (\gamma_1(s), \xi_2 s, \ldots, \xi_n s), \tag{26}
$$

with a function γ_1 yet to be constructed. To this end define $F : \mathbb{R}^2 \to \mathbb{R}$ as

$$
F(\sigma, s) = \frac{d}{dt}\|x(t, t_0, x_0 + \bar{x}) - \mu(t)\|^2\big|_{t=t_0} = 2\langle \bar{x}, f(t_0, x_0 + \bar{x}) - f(t_0, x_0)\rangle, \tag{27}
$$

where $\bar{x} = (\sigma, \xi_2 s, \ldots, \xi_n s)$. Since $f(t_0, x_0 + \bar{x}) = f(t_0, x_0) + D_x f(t_0, x_0)\bar{x} + o(\|\bar{x}\|)$, we find $F(\sigma, s) = \lambda_1 \sigma^2 + s^2 \sum_{i=2}^n \lambda_i \xi_i^2 + o(\sigma^2 + s^2)$. Define

$$
r(\sigma, s) = \begin{cases} \dfrac{F(\sigma, s) - \lambda_1 \sigma^2 - s^2 \sum_{i=2}^n \lambda_i \xi_i^2}{\sigma^2 + s^2} & \text{if } (\sigma, s) \neq (0, 0), \\ 0 & \text{if } (\sigma, s) = (0, 0), \end{cases}
$$

so that r is continuous and, except perhaps at $(0, 0)$, even C^1, and $F(\sigma, s) = \lambda_1 \sigma^2 + s^2 \sum_{i=2}^n \lambda_i \xi_i^2 + r(\sigma, s)(\sigma^2 + s^2)$. Together with (25) this implies that

$$
F(\sigma, s) = (\lambda_1 + r(\sigma, s))\sigma^2 - (\lambda_1 \xi_1^2 - r(\sigma, s))s^2.
$$

Since r is continuous and $r(0, 0) = 0$, there exists $\delta > 0$ such that $|r(\sigma, s)| \leq \min\{\lambda_1, \lambda_1 \xi_1^2\}$ for all $(\sigma, s) \in B_\delta = \{(\sigma, s) : |\sigma| < \delta, |s| < \delta\}$. In order to write

the solutions of $F = 0$ near $(0,0)$ in the form $(\sigma(s), s)$ we define two functions $F^-, F^+ : B_\delta \to \mathbb{R}$ by

$$F^\pm(\sigma, s) = \sqrt{\lambda_1 + r(\sigma, s)}\, \sigma \pm \sqrt{\lambda_1 \xi_1^2 - r(\sigma, s)}\, s \,, \qquad (28)$$

so that $F = F^- F^+$ on B_δ. We first study F^-. Clearly, F^- is C^1 on B_δ except at $(0,0)$, and $F^-(0,0) = 0$. To prove that F^- is actually differentiable at $(0,0)$ with

$$\nabla F^-(0,0) = \left(\sqrt{\lambda_1}, -\sqrt{\lambda_1 \xi_1^2} \right)^\top , \qquad (29)$$

we compute

$$F^-(\sigma, s) - F^-(0,0) - \left(\sqrt{\lambda_1}\, \sigma - \sqrt{\lambda_1 \xi_1^2 - r(\sigma, s)}\, s \right)$$

$$= \left(\sqrt{\lambda_1 + r(\sigma, s)} - \sqrt{\lambda_1} \right) \sigma - \left(\sqrt{\lambda_1 \xi_1^2 - r(\sigma, s)} - \sqrt{\lambda_1 \xi_1^2} \right) s$$

$$= \frac{r(\sigma, s)\sigma}{\sqrt{\lambda_1 + r(\sigma, s)} + \sqrt{\lambda_1}} + \frac{r(\sigma, s)s}{\sqrt{\lambda_1 \xi_1^2 - r(\sigma, s)} + \sqrt{\lambda_1 \xi_1^2}}$$

$$\leq |r(\sigma, s)| \sqrt{\sigma^2 + s^2} \left(\frac{1}{\sqrt{\lambda_1}} + \frac{1}{\sqrt{\lambda_1 \xi_1^2}} \right).$$

Since $r(\sigma, s) \to 0$ as $\sqrt{\sigma^2 + s^2} \to 0$, we deduce that

$$\lim_{\sigma^2 + s^2 \to 0} \frac{|F^-(\sigma, s) - F^-(0,0) - (\sqrt{\lambda_1}\, \sigma - \sqrt{\lambda_1 \xi_1^2 - r(\sigma, s)}\, s)|}{\sqrt{\sigma^2 + s^2}} = 0 \,,$$

which proves (29). A direct computation shows that $\lim_{(\sigma,s) \to (0,0)} \nabla F^-(\sigma, s) = \nabla F^-(0,0)$, hence F^- is a C^1 function. In particular,

$$\frac{\partial}{\partial \sigma} F^-(0,0) = \sqrt{\lambda_1} \neq 0 \,.$$

By the Implicit Function Theorem there exists $\varepsilon > 0$ as well as a C^1 function $\gamma^- : (-\varepsilon, \varepsilon) \to \{\sigma : |\sigma| < \delta\}$ with $\gamma^-(0) = 0$, such that $F^-(\gamma^-(s), s) = 0$ for all $|s| < \varepsilon$. Together with (27) this implies that $(\gamma^-(s), \xi_2 s, \ldots, \xi_n s) \in Z^{\mathrm{nl}}(t_0, x_0)$ for all $s \in (-\varepsilon, \varepsilon)$. Moreover, by (28) we have

$$\dot{\gamma}^-(0) = \lim_{s \to 0} \frac{\gamma^-(s)}{s} = \lim_{s \to 0} \frac{\sqrt{\lambda_1 \xi_1^2 - r(\gamma^-(s), s)}}{\sqrt{\lambda_1 + r(\gamma^-(s), s)}} = \frac{\sqrt{\lambda_1 \xi_1^2}}{\sqrt{\lambda_1}} = |\xi_1| \,.$$

In a completely analogous manner we can also construct, with some $\varepsilon > 0$, a function $\gamma^+ : (-\varepsilon, \varepsilon) \to \{\sigma : |\sigma| < \delta\}$ with $\gamma^+(0) = 0$ such that $(\gamma^+(s), \xi_2 s, \ldots, \xi_n s) \in Z^{\mathrm{nl}}(t_0, x_0)$ for all $s \in (-\varepsilon, \varepsilon)$ and

$$\dot{\gamma}^+(0) = -\frac{\sqrt{\lambda_1 \xi_1^2}}{\sqrt{\lambda_1}} = -|\xi_1| \,.$$

Therefore, by defining

$$\gamma_1(s) := \begin{cases} \gamma^-(s) & \text{if } \xi_1 > 0 \,, \\ \gamma^+(s) & \text{if } \xi_1 < 0 \,, \end{cases}$$

for all $s \in (-\varepsilon, \varepsilon)$ we obtain, via (26), a curve γ with $\gamma(0) = 0$, $\dot{\gamma}(0) = \xi$, and $\mathrm{graph}\,\gamma \subset Z^{\mathrm{nl}}(t_0, x_0)$. This completes the proof. $\qquad \square$

The remaining two results in this section clarify the implications that (quasi)hyperbolicity and ellipticity have on the local behaviour of solutions of (1) on the nonlinear zero strain set.

Theorem 2.13. *(Local Hyperbolicity and Quasihyperbolicity). Consider* $(t_0, x_0) \in$
$I \times \mathbb{R}^n$ *and the solution* μ *of* (1) *with* $\mu(t_0) = x_0$.
(i) If $(t_0, x_0) \in \mathcal{H}_\Gamma$ *then there exists* $\delta > 0$ *such that*

$$\frac{d^2}{dt^2} \|x(t, t_0, x_0 + \bar{x}) - \mu(t)\|_\Gamma^2 \big|_{t=t_0} > 0 \quad \text{for all } \bar{x} \in Z_\Gamma^{\mathrm{nl}}(t_0, x_0) \text{ with } 0 < \|\bar{x}\|_\Gamma < \delta.$$

(ii) If $(t_0, x_0) \in \mathcal{Q}_\Gamma$ *then there exists* $\delta > 0$ *such that*

$$\frac{d^2}{dt^2} \|x(t, t_0, x_0 + \bar{x}) - \mu(t)\|_\Gamma^2 \big|_{t=t_0} < 0 \quad \text{for all } \bar{x} \in Z_\Gamma^{\mathrm{nl}}(t_0, x_0) \text{ with } 0 < \|\bar{x}\|_\Gamma < \delta.$$

Proof. As in the proofs of Theorem 2.11 and Lemma 2.12, it is no restriction to work
with the Euclidean norm, i.e. $\Gamma = \mathrm{id}_{n \times n}$, and to assume that the non-degenerate
matrix $S(t_0, x_0)$ is diagonal, i.e., S is given by (24). By assumption, $S(t_0, x_0)$ is
indefinite, i.e., the eigenvalues λ_i satisfy (7) for an $n^+ \in \{1, \dots, n-1\}$.

To prove (i) assume that $(t_0, x_0) \in \mathcal{H}$. First we will show that for any $\varepsilon > 0$
there exists $\delta > 0$ such that for each $\bar{x} \in Z^{\mathrm{nl}}(t_0, x_0)$ with $0 < \|\bar{x}\| \leq \delta$ we can find
$\xi \in Z(t_0, x_0)$ satisfying

$$\|\xi - \bar{x}\| < \varepsilon \|\bar{x}\|. \tag{30}$$

To this end define $|\lambda| = \min_{1 \leq i \leq n} |\lambda_i|$ and

$$h(\bar{x}) := f(t_0, x_0 + \bar{x}) - f(t_0, x_0) - D_x f(t_0, x_0)\bar{x} = o(\|\bar{x}\|) \quad \text{as } \|\bar{x}\| \to 0.$$

Fix $\varepsilon > 0$ and choose $\delta > 0$ so small that

$$|\langle \bar{x}, h(\bar{x}) \rangle| < \tfrac{1}{2} \varepsilon \lambda \|\bar{x}\|^2 \qquad \text{for all } 0 < \|\bar{x}\| < \delta. \tag{31}$$

Let $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n) \in Z^{\mathrm{nl}}(t_0, x_0)$ with $\|\bar{x}\| \leq \delta$. Then

$$\sum_{i=1}^n \lambda_i \bar{x}_i^2 + \langle \bar{x}, h(\bar{x}) \rangle = 0, \tag{32}$$

and combining (31) and (32) we obtain

$$\sum_{i=1}^{n^+} \lambda_i (1+\varepsilon)\bar{x}_i^2 + \sum_{i=n^++1}^n \lambda_i (1-\varepsilon)\bar{x}_i^2 > 0 \text{ and } \sum_{i=1}^{n^+} \lambda_i (1-\varepsilon)\bar{x}_i^2 + \sum_{i=n^++1}^n \lambda_i (1+\varepsilon)\bar{x}_i^2 < 0.$$

This implies that there exists $\alpha \in (-\varepsilon, \varepsilon)$ such that

$$\xi = (\sqrt{1+\alpha}\,\bar{x}_1, \dots, \sqrt{1+\alpha}\,\bar{x}_{n^+}, \sqrt{1-\alpha}\,\bar{x}_{n^++1}, \dots, \sqrt{1-\alpha}\,\bar{x}_n)^\top \in Z(t_0, x_0).$$

Moreover, using the fact that $|\sqrt{1 \pm \alpha} - 1| \leq |\alpha| < \varepsilon$ we obtain (30).

Since $(t_0, x_0) \in \mathcal{H}$, there exists $\alpha > 0$ such that $\langle \xi, M(t_0, x_0)\xi \rangle > \alpha$ for all
$\|\xi\| = 1$. A direct computation shows that

$$\frac{d^2}{dt^2} \|x(t, t_0, x_0 + \bar{x}) - \mu(t)\|^2 \big|_{t=t_0} = \langle \bar{x}, M(t_0, x_0)\bar{x} \rangle + o(\|\bar{x}\|^2) \quad \text{as } \|\bar{x}\| \to 0,$$

and hence there exists $\delta_1 > 0$ such that for all $\|\bar{x}\| \leq \delta_1$,

$$\left| \frac{d^2}{dt^2} \|x(t, t_0, x_0 + \bar{x}) - \mu(t)\|^2 \big|_{t=t_0} - \langle \bar{x}, M(t_0, x_0)\bar{x} \rangle \right| \leq \tfrac{1}{2} \alpha \|\bar{x}\|^2. \tag{33}$$

Define $m = \|M(t_0, x_0)\|$ as well as

$$\varepsilon = \frac{\alpha}{2\alpha + 6m}, \tag{34}$$

and choose $\delta_2 > 0$ such that (30) holds whenever $0 < \|\bar{x}\| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$ and consider $\bar{x} \in Z^{\mathrm{nl}}(t_0, x_0)$ with $\|\bar{x}\| < \delta$. We have

$$\langle \bar{x}, M(t_0, x_0)\bar{x} \rangle = \langle \xi, M(t_0, x_0)\xi \rangle + \langle \bar{x} - \xi, M(t_0, x_0)\xi \rangle + \langle \bar{x}, M(t_0, x_0)(\bar{x} - \xi) \rangle \,,$$

where $\xi \in Z(t_0, x_0)$ is chosen appropriately such as to satisfy $\|\xi - \bar{x}\| < \varepsilon \|\bar{x}\|$. Together with (33) and (34) this implies that

$$\frac{d^2}{dt^2} \|x(t, t_0, x_0 + \bar{x}) - \mu(t)\|^2 \big|_{t=t_0} > \alpha \|\bar{x}\|^2 - \varepsilon m \|\bar{x}\| \|\xi\| - \varepsilon m \|\bar{x}\|^2 - \tfrac{1}{2}\alpha\|\bar{x}\|^2$$

$$> \tfrac{1}{2}(\alpha - 6\varepsilon m)\|\bar{x}\|^2 = \frac{\alpha^2}{2\alpha + 6m}\|\bar{x}\|^2 > 0 \,.$$

Since the argument for (ii) is completely analogous, the proof is complete. $\qquad \square$

Theorem 2.14 (Local Ellipticity). *Consider $(t_0, x_0) \in I \times \mathbb{R}^n$ and the solution μ of (1) with $\mu(t_0) = x_0$. If $(t_0, x_0) \in \mathcal{E}_\Gamma$ then there exists $\varepsilon > 0$ and two C^1 curves $\gamma^-, \gamma^+ : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ with the following properties:*
(i) $\gamma^\pm(0) = 0$ and $\mathrm{graph}\gamma^\pm \subset Z_\Gamma^{\mathrm{nl}}(t_0, x_0)$.
(ii) For every $x^+ \in \mathrm{graph}\gamma^+ \setminus \{0\}$

$$\frac{d^2}{dt^2} \|x(t, t_0, x_0 + x^+) - \mu(t)\|_\Gamma^2 \big|_{t=t_0} > 0 \,,$$

whereas for every $x^- \in \mathrm{graph}\gamma^- \setminus \{0\}$

$$\frac{d^2}{dt^2} \|x(t, t_0, x_0 + x^-) - \mu(t)\|_\Gamma^2 \big|_{t=t_0} < 0 \,.$$

Proof. As in the proof of Theorem 2.13 we can assume w.l.o.g. that $\Gamma = \mathrm{id}_{n \times n}$ and $S_\Gamma(t_0, x_0)$ is diagonal. Since (t_0, x_0) is elliptic, by Remark 6 there exist two unit vectors $\xi_1, \xi_2 \in Z(t_0, x_0)$ satisfying

$$\langle \xi_1, M(t_0, x_0)\xi_1 \rangle > 0 > \langle \xi_2, M(t_0, x_0)\xi_2 \rangle.$$

Define $\alpha = \min\{\langle \xi_1, M(t_0, x_0)\xi_1 \rangle, -\langle \xi_2, M(t_0, x_0)\xi_2 \rangle\}$ and $m = \|M(t_0, x_0)\|$. According to Lemma 2.12 there exists $\varepsilon_1 > 0$ and two C^1 curves $\gamma^\pm : (-\varepsilon_1, \varepsilon_1) \to \mathbb{R}^n$ satisfying (i), more precisely, $\gamma^\pm(0) = 0$, $\mathrm{graph}\gamma^\pm \subset Z^{\mathrm{nl}}(t_0, x_0)$ and $\dot{\gamma}^+(0) = \xi_1$, $\dot{\gamma}^-(0) = \xi_2$. A direct computation shows that

$$\frac{d^2}{dt^2} \|x(t, t_0, x_0 + \bar{x}) - \mu(t)\|^2 \big|_{t=t_0} = \langle \bar{x}, M(t_0, x_0)\bar{x} \rangle + o(\|\bar{x}\|^2) \quad \text{as } \|\bar{x}\| \to 0 \,.$$

This implies that there exists $\delta > 0$ such that for all $\|\bar{x}\| < \delta$

$$\left| \frac{d^2}{dt^2} \|x(t, t_0, x_0 + \bar{x}) - \mu(t)\|^2 \big|_{t=t_0} - \langle \bar{x}, M(t_0, x_0)\bar{x} \rangle \right| \leq \tfrac{1}{4}\alpha\|\bar{x}\|^2.$$

We now verify (ii) for γ^+. A Taylor-expansion of γ^+ at $s = 0$, using the fact that $\gamma^+(0) = 0$ and $\dot{\gamma}^+(0) = \xi_1$, yields $\gamma^+(s) = \xi_1 s + o(|s|)$. Consequently, there exists $\varepsilon \in (0, \varepsilon_1)$ such that $\|\gamma^+(s) - \xi_1 s\| \leq L|s|$ holds for all $|s| < \varepsilon$; here $L > 0$ is chosen so small that $\tfrac{1}{2}\alpha > \tfrac{1}{4}\alpha(L+1)^2 + mL(L+2)$. Thus

$$\langle \gamma^+(s), M(t_0, x_0)\gamma^+(s) \rangle \geq s^2 \langle \xi_1, M(t_0, x_0)\xi_1 \rangle - |\langle \gamma^+(s) - \xi_1 s, M(t_0, x_0)s\xi_1 \rangle|$$

$$- |\langle \gamma^+(s), M(t_0, x_0)(\gamma^+(s) - s\xi_1) \rangle|$$

$$\geq s^2 \big(\alpha - mL(L+2)\big)$$

whenever $0 < |s| < \varepsilon$, and therefore

$$\frac{d^2}{dt^2}\|x(t,t_0,x_0+\gamma^+(s)) - \mu(t)\|^2\big|_{t=t_0} \geq s^2\big(\alpha - mL(L+2)\big) - \tfrac{1}{4}\alpha(L+1)^2$$
$$> \tfrac{1}{2}\alpha s^2 > 0\,.$$

As the verification of (ii) for γ^- is completely analogous, the proof is complete. $\qquad\square$

2.3. **An algorithm to compute the dynamic partition.** In this section we describe an algorithm to compute the dynamic partition associated with (1) with respect to the norm $\|\cdot\|_\Gamma = \langle\cdot,\Gamma\cdot\rangle^{\frac{1}{2}}$. Given $(t_0,x_0) \in I\times\mathbb{R}^n$, the following schematic algorithm lists the steps required for determining the type of (t_0,x_0).

```
Simplify Norm
Diagonalize Strain
begin switch type of S̃ = (n⁺, n⁻) satisfies
      n⁺ + n⁻ < n, then (t₀, x₀) is degenerate
      n⁺ + n⁻ = n, then
      begin switch n⁺n⁻ is
         zero then
            begin switch n⁺ is
               zero, then (t₀, x₀) is attracting
               non-zero, then (t₀, x₀) is repelling
            end switch
         non-zero then
            begin
               Simplify Zero Strain Set
               Construct Polynomial
               begin switch rangeP contains
                  only positive values then (t₀, x₀) is hyperbolic
                  only negative values then (t₀, x₀) is quasihyperbolic
                  positive and negative values then (t₀, x₀) is elliptic
               end switch
            end
      end switch
   end switch
end switch
```

All steps in the above algorithm will now be explained and justified in detail.

Simplify norm. Choose $T \in \mathbb{R}^{n\times n}$ such that $T^\top\Gamma T = \mathrm{id}_{n\times n}$, i.e., T^{-1} is a root of Γ. Apply the transformation $x \mapsto T^{-1}x$ to

$$\dot{x} = f(t,x) \qquad \text{with norm } \|\cdot\|_\Gamma = \langle\cdot,\Gamma\cdot\rangle^{\frac{1}{2}}. \tag{35}$$

Lemma 2.7 implies that (t_0,x_0) for (35) and $(t_0,T^{-1}x_0)$ for

$$\dot{x} = T^{-1}f(t,Tx) \qquad \text{with Euclidean norm } \|\cdot\| = \langle\cdot,\cdot\rangle^{\frac{1}{2}} \tag{36}$$

have the same type. By (15) the strain tensor and strain acceleration tensor for (36) at $(t_0,T^{-1}x_0)$ are, respectively, given by

$$S = T^\top S_\Gamma(t_0,x_0)T \qquad \text{and} \qquad M = T^\top M_\Gamma(t_0,x_0)T.$$

Diagonalize strain. Choose an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that

$$\widetilde{S} = Q^\top SQ = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)\,,$$

with the eigenvalues λ_i ordered in accordance with (7). Apply the transformation $x \mapsto Q^\top x$ to (36). By Lemma 2.5, the type does not change under this transformation, and $\widetilde{S}$ and $\widetilde{M} = Q^\top M Q$ are the new strain and strain acceleration tensors, respectively. Thus (t_0, x_0) is degenerate if $n^+ + n^- < n$. If $n^+ + n^- = n$ and $n^+ n^- = 0$ then (t_0, x_0) is attracting or repelling, depending on whether $n^+ = 0$ or $n^+ \neq 0$. In the remaining case, that is, if $n^+ + n^- = n$ and $n^+ n^- \neq 0$, then $\widetilde{S}$ is indefinite and non-degenerate, and to determine whether (t_0, x_0) is elliptic, hyperbolic or quasihyperbolic we simplify the zero strain set.

Simplify zero strain set. Define the matrices $N := \mathrm{diag}(1/\sqrt{|\lambda_1|}, \ldots, 1/\sqrt{|\lambda_n|})$ and

$$\widehat{M} = \left(\widehat{m}_{ij}\right)_{ij} := N \widetilde{M} N \quad \text{with} \quad \widehat{m}_{ij} = \frac{\widetilde{m}_{ij}}{\sqrt{|\lambda_i \lambda_j|}},$$

as well as the set

$$\widehat{Z} := \left\{\xi \in \mathbb{R}^n : \xi_1^2 + \ldots + \xi_{n^+}^2 = \xi_{n^++1}^2 + \ldots + \xi_n^2\right\}.$$

By Remark 2, $N\xi \in \widetilde{Z}$ if and only if $\xi \in \widehat{Z}$. Moreover, $\langle N\xi, \widetilde{M} N\xi \rangle = \langle \xi, \widehat{M}\xi \rangle$ and thus $\widetilde{M}|_{\widetilde{Z}}$ is positive/negative definite or indefinite precisely if $\widehat{M}|_{\widehat{Z}}$ has the same property.

Construct polynomial (characterizing strain acceleration on zero strain set). Since the sign of $\langle \xi, \widehat{M}\xi \rangle$ does not depend on the norm $\|\xi\| \neq 0$, we restrict our attention to $\xi \in \widehat{Z}$ with

$$\xi_1^2 + \ldots + \xi_{n^+}^2 = \xi_{n^++1}^2 + \ldots + \xi_n^2 = 1, \tag{37}$$

i.e. $\|\xi\| = \sqrt{2}$, to check whether $\widehat{M}|_{\widehat{Z}}$ is positive/negative definite or indefinite. Moreover, if n_+ equals 1 or $n-1$ then w.l.o.g. we can set $\xi_1 = 1$ or $\xi_n = 1$ in (37), respectively. Let

$$B_{k,\ell} = \{(\xi_k, \xi_{k+1}, \ldots, \xi_\ell)^\top \in \mathbb{R}^{\ell-k+1} \mid \xi_k^2 + \xi_{k+1}^2 + \ldots + \xi_\ell^2 = 1\}$$

denote the $(l - k)$-dimensional unit sphere, where $\ell - k \geq 1$. We parametrize $B_{k,\ell}$ by $\ell - k$ angle variables as follows:

$$\begin{aligned}
\xi_k &= \cos \alpha_k, \\
\xi_{k+1} &= \sin \alpha_k \cos \alpha_{k+1}, \\
&\vdots \\
\xi_{\ell-1} &= \sin \alpha_k \sin \alpha_{k+1} \cdot \ldots \cdot \sin \alpha_{\ell-2} \cos \alpha_{\ell-1}, \\
\xi_\ell &= \sin \alpha_k \sin \alpha_{k+1} \cdot \ldots \cdot \sin \alpha_{\ell-2} \sin \alpha_{\ell-1},
\end{aligned} \tag{38}$$

where $\alpha_i \in [0, 2\pi]$ for all $k \leq i \leq \ell - 1$. Letting $z_i = \tan \frac{\alpha_i}{2}$ and using the fact that, consequently, for all $k \leq i \leq \ell - 1$, $\cos \alpha_i = \frac{1-z_i^2}{1+z_i^2}$ as well as $\sin \alpha_i = \frac{2z_i}{1+z_i^2}$, we obtain from (38) that

$$\xi_i = \begin{cases} \frac{1-z_i^2}{1+z_i^2} \prod_{j=k}^{i-1} \frac{2z_j}{1+z_j^2} & \text{if } k \leq i \leq \ell - 1, \\ \prod_{j=k}^{\ell-1} \frac{2z_j}{1+z_j^2} & \text{if } i = \ell, \end{cases} \tag{39}$$

where we use the convention that $\prod_{j=k}^{k-1} \frac{2z_j}{1+z_j^2} = 1$. Next we utilize (37) and (39) to construct a polynomial which characterizes the sign of $\langle \xi, \widehat{M}\xi \rangle$ for $\xi \in \widehat{Z}$. In

accordance with (37) we check the sign of $\langle \xi, \widehat{M}\xi \rangle$ for

$$\xi = (\xi_1, \ldots, \xi_n)^\top \in \widehat{Z} \quad \text{with} \quad \|\xi\| = \sqrt{2}, \tag{40}$$

depending on n^+. If $n^+ = 2, \ldots, n - 2$ then

$$(\xi_1, \ldots, \xi_{n^+})^\top \in B_{1,n^+} \quad \text{and} \quad (\xi_{n^++1}, \ldots, \xi_n)^\top \in B_{n^++1,n}. \tag{41}$$

If n^+ equals 1 or $n - 1$ then w.l.o.g. $\xi_1 = 1$ or $\xi_n = 1$ in (37), respectively. For each of these different cases we construct a polynomial P which allows us to decide whether $\widehat{M}_{\widehat{Z}}$ is positive/negative definite or indefinite.

Case $n^+ = 1$: Assume w.l.o.g. that $\xi_1 = 1$. If $n = 2$ then (37) takes the form $\xi_1^2 = \xi_2^2$ and we need to deal with this case separately.

Subcase $n = 2$: Since $\xi_1 = 1$,

$$\langle \xi, \widehat{M}\xi \rangle = \widehat{m}_{11} + 2\widehat{m}_{12}\xi_2 + \widehat{m}_{22}\xi_2^2.$$

Replacing ξ_2 by z and using the fact that $\xi_2^2 = 1$ we get the polynomial

$$P(z) = \widehat{m}_{22}z^2 + 2\widehat{m}_{12}z + \widehat{m}_{11} \quad \text{with} \quad z \in \{-1, 1\},$$

and $\text{sign}\langle \xi, \widehat{M}\xi \rangle = \text{sign}P(z)$, i.e. (t_0, x_0) is hyperbolic/quasihyperbolic if

$$\text{range}P = \big\{ P(z) : z \in \{-1, 1\} \big\}$$

contains only positive/negative values and elliptic if it contains both positive and negative values.

Subcase $n \geq 3$: Since $\xi_1 = 1$,

$$\langle \xi, \widehat{M}\xi \rangle = \sum_{i,j=2}^{n} \widehat{m}_{ij}\xi_i\xi_j + 2\sum_{i=2}^{n} \widehat{m}_{1i}\xi_i + \widehat{m}_{11} \tag{42}$$

with (40), i.e. $(\xi_2, \ldots, \xi_n)^\top \in B_{2,n}$. Substituting (39) into (42) yields a rational function, and multiplying this function with its denominator

$$(1 + z_2^2)^2 \cdot \ldots \cdot (1 + z_{n-1}^2)^2,$$

we obtain the following polynomial P of degree $4(n - 2)$,

$$P(z) = \sum_{i,j=2}^{n} \widehat{m}_{ij}P_{ij}(z) + 2\sum_{i=2}^{n} \widehat{m}_{1i}Q_i(z) + \widehat{m}_{11}R(z), \tag{43}$$

where

$$R(z) = (1 + z_2^2)^2 \cdot \ldots \cdot (1 + z_{n-1}^2)^2,$$

and $P_{ij}(z), Q_i(z), 2 \leq i, j \leq n$ are polynomials of the form

$$P_{ij}(z) = P_i(z)P_j(z) \quad \text{and} \quad Q_i(z) = (1 + z_2^2) \cdot \ldots \cdot (1 + z_{n-1}^2)P_i(z)$$

with

$$P_i(z) = \begin{cases} (1 - z_i^2) \prod_{j=2}^{i-1} 2z_j \prod_{j=i+1}^{n-1}(1 + z_j^2) & \text{if } 2 \leq i < n, \\ \prod_{j=2}^{n-1} 2z_j & \text{if } i = n. \end{cases}$$

Then (t_0, x_0) is hyperbolic/quasihyperbolic if

$$\text{range}P = \{P(z) : z = (z_2, \ldots, z_{n-1})^\top \in \mathbb{R}^{n-2}\}$$

contains only positive/negative values, and elliptic if $\text{range}P$ contains both positive and negative values.

Case $n^+ \in \{2, \ldots, n-2\}$: To check the sign of $\langle \xi, \widehat{M}\xi \rangle$ for ξ satisfying (41) we define $z = (z_1, \ldots, z_{n^+-1}, z_{n^++1}, \ldots, z_{n-1})$ and substitute (39) into $\langle \xi, \widehat{M}\xi \rangle$. As before, this yields a rational function which we again multiply with its denominator

$$(1 + z_1^2)^2 \cdot \ldots \cdot (1 + z_{n^+-1}^2)^2 \cdot (1 + z_{n^++1}^2)^2 \cdot \ldots \cdot (1 + z_{n-1}^2)^2$$

to obtain a polynomial P of degree $4(n-2)$,

$$P(z) = \sum_{i,j=1}^{n} \widehat{m}_{ij} P_{ij}(z),$$

where each $P_{ij}(z)$, $1 \leq i, j \leq n$, is a polynomial of the form $P_{ij}(z) = P_i(z)P_j(z)$ with

$$P_i(z) = \begin{cases} (1 - z_i^2) \prod_{j=1}^{i-1} 2z_j \prod_{j=i+1}^{n^+-1}(1 + z_j^2) \prod_{j=n^++1}^{n-1}(1 + z_j^2) \\ \qquad\qquad\qquad\qquad\qquad \text{for } 1 \leq i \leq n^+ - 1, \\[2mm] \prod_{j=1}^{n^+-1} 2z_j \prod_{j=n^++1}^{n-1}(1 + z_j^2) \qquad\qquad \text{for } i = n^+, \\[2mm] (1 - z_i^2) \prod_{j=n^++1}^{i-1} 2z_j \prod_{j=i+1}^{n-1}(1 + z_j^2) \prod_{j=1}^{n^+-1}(1 + z_j^2) \\ \qquad\qquad\qquad\qquad\qquad \text{for } n^+ + 1 \leq i \leq n - 1, \\[2mm] \prod_{j=n^++1}^{n-1} 2z_j \prod_{j=1}^{n^+-1}(1 + z_j^2) \qquad\qquad \text{for } j = n. \end{cases}$$

Consequently, (t_0, x_0) is hyperbolic/quasihyperbolic if

$$\mathrm{range} P = \{P(z) : z = (z_1, \ldots, z_{n^+-1}, z_{n^++1}, \ldots, z_{n-1}) \in \mathbb{R}^{n-2}\}$$

contains only positive/negative values, and elliptic if $\mathrm{range} P$ contains both positive and negative values.

Case $n^+ = n - 1$: Note that only the case $n \geq 3$ has to be considered here, since $n = 2$ implies $n^+ = 1$, which case has been discussed earlier. Assume w.l.o.g. that $\xi_n = 1$ so that

$$\langle \xi, \widehat{M}\xi \rangle = \sum_{i,j=1}^{n-1} \widehat{m}_{ij}\xi_i\xi_j + 2\sum_{i=1}^{n-1} \widehat{m}_{in}\xi_i + \widehat{m}_{nn}, \qquad (44)$$

where $(\xi_1, \ldots, \xi_{n-1})^\top \in B_{1,n-1}$. We define $z = (z_1, \ldots, z_{n-2})$ and substitute (39) into (44). The result is again a rational function, and multiplying this function with its denominator

$$(1 + z_1^2)^2 \cdot \ldots \cdot (1 + z_{n-2}^2)^2$$

we obtain the polynomial P of degree $4(n-2)$,

$$P(z) = \sum_{i,j=1}^{n-1} \widehat{m}_{ij} P_{ij}(z) + 2\sum_{i=1}^{n-1} \widehat{m}_{in} Q_i(z) + \widehat{m}_{nn} R(z), \qquad (45)$$

where

$$R(z) = (1 + z_1^2)^2 \cdot \ldots \cdot (1 + z_{n-2}^2)^2,$$

and $P_{ij}(z), Q_i(z), 1 \leq i, j \leq n - 1$ are polynomials of the form

$$P_{ij}(z) = P_i(z)P_j(z) \text{ and } Q_i(z) = (1 + z_1^2) \cdot \ldots \cdot (1 + z_{n-2}^2)P_i(z)$$

with

$$P_i(z) = \begin{cases} (1 - z_i^2) \prod_{j=1}^{i-1} 2z_j \prod_{j=i+1}^{n-2}(1 + z_j^2) & \text{if } 1 \le i < n-1 \,, \\ \prod_{j=1}^{n-2} 2z_j & \text{if } i = n-1 \,. \end{cases}$$

Thus (t_0, x_0) is hyperbolic/quasihyperbolic if

$$\text{range} P = \{P(z) : z = (z_1, \ldots, z_{n-2}) \in \mathbb{R}^{n-2}\}$$

contains only positive/negative values, and elliptic if $\text{range} P$ contains both positive and negative values.

Example 1. The polynomial P which characterizes the strain acceleration on the zero strain set in three dimensions, i.e. for $x \in \mathbb{R}^3$, is of the form (see (43) and (45))

$$P(z) = A_4 z^4 + A_3 z^3 + A_2 z^2 + A_1 z + A_0 \quad (z \in \mathbb{R}) \,,$$

where the coefficients $A_i, 0 \le i \le 4$, are determined as follows: If $n^+ = 1$ then

$$\begin{aligned} A_4 &= \widehat{m}_{11} - 2\widehat{m}_{12} + \widehat{m}_{22} \,, \\ A_3 &= 4\widehat{m}_{13} - 4\widehat{m}_{23} \,, \\ A_2 &= 2\widehat{m}_{11} - 2\widehat{m}_{22} + 4\widehat{m}_{33} \,, \\ A_1 &= 4\widehat{m}_{13} + 4\widehat{m}_{23} \,, \\ A_0 &= \widehat{m}_{11} + 3\widehat{m}_{12} + \widehat{m}_{22} \,, \end{aligned}$$

and if $n^+ = 2$ then

$$\begin{aligned} A_4 &= \widehat{m}_{11} - 2\widehat{m}_{13} + \widehat{m}_{33} \,, \\ A_3 &= -4\widehat{m}_{12} + 4\widehat{m}_{23} \,, \\ A_2 &= -2\widehat{m}_{11} + 4\widehat{m}_{22} + 2\widehat{m}_{33} \,, \\ A_1 &= 4\widehat{m}_{12} + 4\widehat{m}_{23} \,, \\ A_0 &= \widehat{m}_{11} + 2\widehat{m}_{13} + \widehat{m}_{33} \,, \end{aligned}$$

cf. [10] for a similar but slightly different formula. If $|z| > \max\left\{1, \frac{|A_0| + |A_1| + |A_2| + |A_3|}{|A_4|}\right\}$ then $|P(z)| > 0$, since

$$\begin{aligned} |P(z)| &\ge |A_4||z|^4 - |A_3||z|^3 - |A_2||z|^2 - |A_1||z| - |A_0| \\ &\ge |A_4||z|^4 - (|A_3| + |A_2| + |A_1| + |A_0|)|z|^3 \\ &= |z|^3 \big(|A_4||z| - (|A_3| + |A_2| + |A_1| + |A_0|)\big) > 0 \,. \end{aligned}$$

Thus we can easily estimate the compact domain which contains all possible zeros of P. It is worth pointing out that for $x \in \mathbb{R}^n$ with $n \ge 4$ the polynomial P depends on $z \in \mathbb{R}^{n-2}$. While it is possible to decide in polynomial time whether either $|P(z)| > 0$ for all $z \in \mathbb{R}^{n-2}$ or else $P(z_1) > 0 > P(z_2)$ for some z_1, z_2, such a decision can be a practically hard problem already for $n = 6$, see e.g. [1].

2.4. **An example.** In this final section we compute the dynamic partition for the equation

$$\begin{aligned} \dot{x} &= (x^3 - x)(y^2 - 1)z \,, \\ \dot{y} &= (x^2 - 1)(y^3 - y)z \,, \\ \dot{z} &= (x^2 + y^2 - \tfrac{1}{4})(z^2 - 1) \,, \end{aligned} \quad (46)$$

for $(x, y, z) \in [-1, 1]^3 \subset \mathbb{R}^3$ with respect to the Euclidean norm. The faces of the cube $[-1, 1]^3$ are invariant under the flow. All horizontal edges $\{(x, y, z) : |z| =$
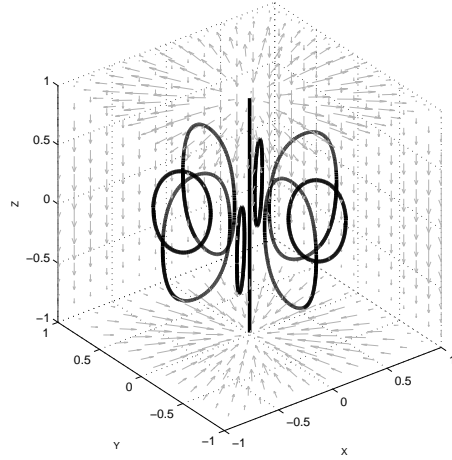
FIGURE 4. The vector field of (46) and some trajectories.

$1, (|x| - 1)(|y| - 1) = 0\}$ consist of equilibria. A heteroclinic orbit connects the two equilibria $(0, 0, -1)$ and $(0, 0, 1)$, see Figure 4. Since (46) is autonomous, by Remark 5, its dynamic partition is independent of time, i.e., the regions $\mathcal{T}(t) \equiv \mathcal{T}$ for $\mathcal{T} \in \{\mathcal{A}, \mathcal{R}, \mathcal{E}, \mathcal{H}, \mathcal{Q}, \mathcal{D}\}$ as well as S, Z and M do not depend on t. For example, the components of the strain tensor $S(x, y, z) = (s_{ij})_{i,j=1,2,3}$ at a point $(x, y, z) \in [-1, 1]^3$ are

$$s_{11} = (3x^2 - 1)(y^2 - 1)z, \quad s_{22} = (x^2 - 1)(3y^2 - 1)z, \quad s_{33} = 2(x^2 + y^2 - \tfrac{1}{4})z,$$

$$s_{12} = s_{21} = xyz(x^2 + y^2 - 2), \quad s_{13} = s_{31} = \tfrac{1}{2}x(x^2y^2 - x^2 - y^2 + 2z^2 - 1),$$

and

$$s_{23} = s_{32} = \tfrac{1}{2}y(x^2y^2 - x^2 - y^2 + 2z^2 - 1).$$

An easy but lengthy computation shows that a point (x, y, z) is attracting if and only if

$$s_{11} + s_{22} + s_{33} \quad < \quad 0,$$
$$s_{11}s_{22} + s_{11}s_{33} + s_{22}s_{33} - s_{12}s_{21} - s_{13}s_{31} - s_{23}s_{32} \quad > \quad 0,$$
$$s_{11}s_{22}s_{33} - s_{11}s_{23}s_{32} - s_{22}s_{13}s_{31} - s_{33}s_{12}s_{21} + 2s_{12}s_{23}s_{31} \quad < \quad 0,$$

and it is repelling if and only if

$$s_{11} + s_{22} + s_{33} \quad > \quad 0,$$
$$s_{11}s_{22} + s_{11}s_{33} + s_{22}s_{33} - s_{12}s_{21} - s_{13}s_{31} - s_{23}s_{32} \quad > \quad 0,$$
$$s_{11}s_{22}s_{33} - s_{11}s_{23}s_{32} - s_{22}s_{13}s_{31} - s_{33}s_{12}s_{21} + 2s_{12}s_{23}s_{31} \quad > \quad 0.$$

The points satisfying either of these two sets of polynomial inequalities define the open interior of a solid torus, see Figure 5. We use the algorithm described in Section 2.3 to compute the elliptic region $\mathcal{E}$ in $[-1, 1]^3$. Figure 6 shows the boundary of the elliptic region (in red), as well as some yellow points indicating on which side of the boundary the elliptic points are actually located. In fact, the plane $z = 0$ consists of degenerate points, since
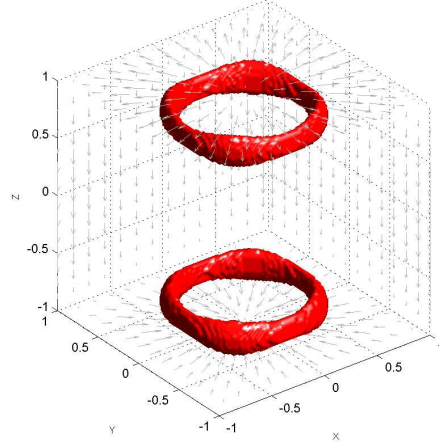
FIGURE 5.   The lower torus consists of attracting points, and the upper torus contains all repelling points of (46) in $[-1, 1]^3$.

$$S(x, y, 0) = \begin{pmatrix} 0 & 0 & s_{13} \\ 0 & 0 & s_{23} \\ s_{31} & s_{32} & 0 \end{pmatrix}$$

is degenerate. To better visualize the elliptic region we do not display the square $\{(x, y, z) \in [-1, 1]^3 : z = 0\}$ belonging to the boundary of $\mathcal{E}$ so as not to hide the structures underneath. Figure 6 again shows the top and bottom torus consisting
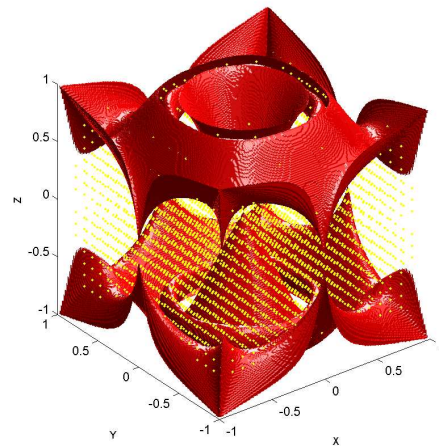

FIGURE 6.   Part of the boundary of the elliptic region $\mathcal{E}$ of (46) in $[-1, 1]^3$ (in red; the square $\{(x, y, z) \in [-1, 1]^3 : z = 0\} \subset \mathcal{D}$ is not displayed). $\mathcal{E}$ is indicated by yellow diamonds.

of repelling and attracting points, respectively. Outside the tori, the elliptic region is indicated by yellow diamonds.

The remaining boundary of $\mathcal{E}$ consists of two cones which meet at the origin and a tent-like structure which altogether form the boundary to the hyperbolic region. The results of the computation of the hyperbolic region $\mathcal{H}$ in $[-1, 1]^3$ are displayed in Figure 7. In agreement with Theorem 2.9, the hyperbolic rest points $(0, 0, -1)$ and $(0, 0, 1)$ together with cone-shaped neighbourhoods thereof are contained in $\mathcal{H}$.
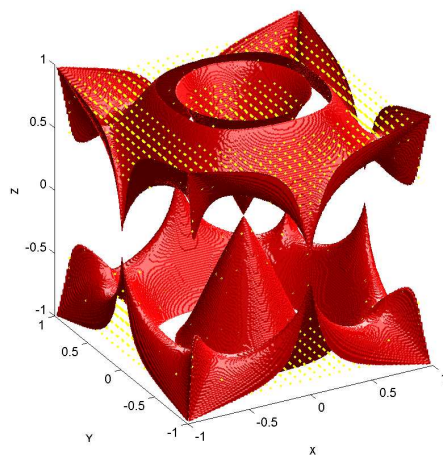

FIGURE 7. The boundary of $\mathcal{H}$ of (46) in $[-1, 1]^3$ (in red). The hyperbolic region $\mathcal{H}$ itself is indicated by yellow diamonds.

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