ON ALMOST AUTOMORPHIC DYNAMICS IN SYMBOLIC LATTICES

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ABSTRACT. We study the existence, structure, and topological entropy of almost automorphic arrays in symbolic lattice dynamical systems. In particular we show that almost automorphic arrays with arbitrarily large entropy are typical in symbolic lattice dynamical systems. Applications to pattern formation and spatial chaos in infinite dimensional lattice systems are considered, and the construction of chaotic almost automorphic signals is discussed.

1. INTRODUCTION

Almost automorphic functions were introduced by Bochner in 1955 ([5]). These functions generalize the (Bohr) almost periodic functions, and they are known to coincide with the Levitan class of N-almost periodic functions ([28]). The closure of an almost automorphic orbit in a dynamical system is minimal ([41]), and an almost automorphic function yielding a compact flow is necessarily uniformly continuous. Every almost periodic function is almost automorphic, but the converse does not hold in general. Almost automorphic functions defined on a locally compact Abelian group and valued in a Banach space admit well-defined Fourier series. But unlike for the almost periodic case, Fourier series of an almost automorphic function need not be unique, and the respective Bochner-Fejer sums only converge pointwise in general ([41]). Nevertheless, the frequency module of an almost automorphic function, defined as the smallest additive group containing the Fourier spectrum, is well-defined and algebraically isomorphic to the dual group of the maximal almost periodic factor of the hull of the almost automorphic function in question ([39]). It is due to these harmonic properties that both almost periodic and almost automorphic functions may be considered natural generalizations of periodic functions in a stronger and a weaker sense, respectively.

Almost automorphic functions and flows play an important role in characterizing recurrence, randomness and complexity of dynamical systems. They are known to be fundamental in almost periodically forced differential equations, simply because almost automorphic solutions largely exist in such systems but almost periodic ones need not ([21, 22, 37, 39]). The complexity of almost automorphic dynamics has mainly been studied for symbolic flows. It was first observed in [13, 32] that almost automorphic symbolic flows can differ quite considerably from periodic and almost

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periodic ones, e.g., by exhibiting positive entropy and/or lacking unique ergodicity. Following the work of [20, 31, 33, 43] on the characterization of almost automorphic symbolic sequences, Toeplitz sequences have received considerable attention recently as a special class of almost automorphic sequences (e.g. [2, 10, 11, 15, 19]). While regular Toeplitz sequences are uniquely ergodic with zero topological entropy, it is known that irregular Toeplitz sequences are typically not uniquely ergodic and exhibit positive topological entropy, though they may also be uniquely ergodic with positive entropy ([15, 19]).

The aim of the present paper is to study almost automorphic dynamics in higher dimensional symbolic flows, i.e., symbolic lattice systems $(S^{\mathbb{Z}^d}, \mathbb{Z}^d)$, where $S = \{s_0, s_1, \ldots, s_{m-1}\}$ denotes a finite alphabet and a $\mathbb{Z}^d$-action is defined via the full shift of symbols. By generalizing results for 1D systems, not only will a complete characterization of almost automorphic arrays including Toeplitz arrays be given, but also their chaotic nature will be analyzed in terms of the positivity of topological entropy. In fact, similar to the 1D case with two symbols ([43]), we will show that typical irregular almost automorphic arrays lack unique ergodicity and have topological entropy arbitrarily close to the maximal entropy of the underlying symbolic lattice system.

The main motivation for this study comes from the role played by almost automorphic symbolic lattice flows in pattern formation and the onset of spatial chaos in high (spatial) dimensional lattice dynamical systems. Pattern formation and spatial chaos have been studied extensively in lattice dynamical systems during recent years, mainly due to important potential applications in chemistry, image processing, biology, and communications (see [1, 3, 8, 9, 24, 40] and references therein). Since almost automorphic dynamics is close to periodic dynamics but at the same time can be complicated and even chaotic, it is to be expected that almost automorphy may be responsible for the formation of regular patterns as well as the existence of spatial chaos in lattice dynamical systems. The present study gives support to this assertion.

Further motivation for this study stems from the problem of constructing almost automorphic chaotic signals. Since chaotic signals, which are non-periodic yet short-term predictable, combine in a sense determinism and randomness, they have repeatedly been proposed as information-bearing signals for secure communication in tele-communication and information technology. Various methods of constructing chaotic signals have been proposed in recent years, most of them based on symbolic dynamics and solutions of chaotic systems such as Lorenz's equations and Chua's circuit (see [29, 34] and references therein). Our study on symbolic flows provides an alternative approach to the generation of chaotic signals which, due to their almost automorphic properties, exhibit a somewhat more visible structure.

This paper is organized as follows. Section 2 contains basic definitions as well as a simple yet useful inheritance lemma. In Section 3 almost automorphic symbolic lattices are constructed and their most important dynamical properties are studied. Applications of the general results to toral rotations and Toeplitz sequences are discussed in some detail. Section 4 demonstrates the relevance of the results obtained earlier to the fascinating topic of pattern formation and spatial chaos in (continuous-time) lattice dynamical systems. Finally, in Section 5 a simple method of constructing almost automorphic signals is presented together with a number of significant examples.

2. Preliminaries

2.1. **Almost automorphy.** Let T be a locally compact Abelian group and X a complete metric space.

Definition 2.1. (i) A continuous function $f : T \to X$ is said to be *almost automorphic* if whenever t_α is a sequence such that $f(t+t_\alpha) \to g(t)$ holds pointwise for some function g, then also $g(t-t_\alpha) \to f(t)$.

(ii) A compact flow (X,T) is *almost automorphic minimal* if it is the closure of an almost automorphic orbit.

We define the *frequency module* of an almost automorphic minimal set as the dual group of its maximal almost periodic factor. This definition makes sense because the phase space of an almost periodic minimal flow is a topological group, and all maximal almost periodic factors of a minimal flow are isomorphic. In [41] the structure of an almost automorphic minimal flow is characterized as follows.

Lemma 2.1 (Veech's structure theorem)**.** *A minimal flow (X,T) is almost automorphic if and only if it is an almost* 1-1 *extension of its maximal almost periodic factor (Y,T), i.e., there is a residual subset $Y_0 \subset Y$ such that each fiber over Y_0 is a singleton.*

Almost automorphic minimal flows admit the following inheritance property which is similar to the corresponding property for minimal flows ([12]).

Lemma 2.2 (Inheritance Property)**.** *Let S be a syndetic subgroup of T. Then (X,T) is almost automorphic minimal with maximal almost periodic factor (Y,T) if and only if (X,S) is almost automorphic minimal with maximal almost periodic factor (Y,S). Consequently, (X,T) is almost periodic if and only if (X,S) is.*

Proof. Since S is a syndetic subgroup of T, there is a compact set $K \subset T$ such that $S+K=T$. If $x_0 \in X$ is an almost automorphic point of (X,T), then it is clear that it is also an almost automorphic point of (X,S). Conversely, let $x_0 \in X$ be an almost automorphic point of (X,S) and let t_α be a sequence in T such that $x_0 \cdot t_\alpha \to x_*$. Write $t_\alpha = s_\alpha + k_\alpha$, where $s_\alpha \in S, k_\alpha \in K$. Without loss of generality, we assume that $x_0 \cdot s_\alpha \to x_1$, $k_\alpha \to k \in K$. Then $x_* = x_1 \cdot k$ and hence

$$x_* \cdot (-t_\alpha) = x_1 \cdot k \cdot (-t_\alpha) = x_1 \cdot (-s_\alpha) \cdot (k - k_\alpha) \to x_0 \cdot 0 = x_0\,.$$

This shows that x_0 is also an almost automorphic point of (X,T). By [12], the orbit closure of x_0 equals X under both the actions of T and S.

The assertion about the maximal almost periodic factor follows immediately from Lemma 2.1. It also follows from Lemma 2.1 that a minimal flow (X,T) is almost periodic if and only if it is almost automorphic minimal and every point in X is almost automorphic. Hence, (X,T) is almost periodic if and only if (X,S) is. □

A typical application of the above lemma is the suspension of continuous maps. Let f denote a homeomorphism of the compact metric space (X,d). The *suspension space* X^f is obtained from $[0,1] \times X$ by identifying for all $x \in X$ the points $(1,x)$ and $\big(0,f(x)\big)$. This space is itself a compact metric space (see e.g. [25]). We denote by d^f any metric inducing the topology of X^f and assume that d^f and d coincide on $\{0\} \times X \subset X^f$. As a continuous-time interpolation of f, the *suspension flow*

$(X^f, \mathbb{R})$ is defined by making points travel along the first coordinate with constant velocity, more explicitly

$$(a, x) \cdot t = \big(a + t - \lfloor a + t \rfloor, f^{\lfloor a+t \rfloor}(x)\big)$$

for all $(a, x) \in X^f$ and $t \in \mathbb{R}$; here and throughout, $\lfloor r \rfloor$ denotes the integral part of any $r \in \mathbb{R}$. Applying Lemma 2.2, we immediately obtain

Proposition 2.1. *The discrete flow (X, f) is almost automorphic (almost periodic, periodic, uniquely ergodic) if and only if the suspension flow $(X^f, \mathbb{R})$ has the corresponding property. Moreover, both flows have the same topological entropy.*

As a special case, the Denjoy flow on the 2-torus is almost automorphic minimal with topological entropy equal to zero. We note that as a subflow of a fixed-point-free C^1 flow (in continuous time) on the 2-torus, the Denjoy flow is topologically conjugate to a suspension of a Denjoy homeomorphism on the circle which in turn is an almost 1-1 extension of a pure rotation ([30]) and therefore almost automorphic (in discrete time). An argument directly using the definition of almost automorphy shows that Aubry-Mather sets on the annulus are also almost automorphic (with vanishing topological entropy), and so are their suspensions.

2.2. **Symbolic lattice systems.** Let $S = \{s_0, s_1, \dots, s_{m-1}\}$ be an alphabet of $m \geq 2$ symbols endowed with a metric d and let $\Sigma = S^{\mathbb{Z}^d}$ be the set of all bi-infinite $\mathbb{Z}^d$-arrays furnished with the product topology. With this topology Σ is compact and metrizable. For instance, if $x = \{x(k)\}_{k \in \mathbb{Z}^d}, y = \{y(k)\}_{k \in \mathbb{Z}^d} \in \Sigma$, then for any $\beta > 1$,

$$\begin{aligned} d_1(x, y) &:= \sum_{k \in \mathbb{Z}^d} \beta^{-|k|} d\big(x(k), y(k)\big), \\ d_2(x, y) &:= \beta^{-\kappa}, \\ d_3(x, y) &:= \sum_{k \in \mathbb{Z}^d} \beta^{-|k|} \frac{d\big(x(k), y(k)\big)}{1 + d\big(x(k), y(k)\big)} \end{aligned}$$

are metrics inducing the product topology on Σ, where, for $k = (k_1, k_2 \cdots, k_d) \in \mathbb{Z}^d$, $|k| = \max\{|k_j|, 1 \leq j \leq d\}$, and $\kappa = \max\{m \in \mathbb{N} : x(k) = y(k), |k| < m\}$. (Here and in the sequel $\mathbb{N}$ denotes the set of positive integers.)

We define the *symbolic d-lattice dynamical system* $(\Sigma, \mathbb{Z}^d)$ as the full shift, that is, $\{x(k)\} \cdot l = \{x(k + l)\}$ for all $\{x(k)\} \in \Sigma, l \in \mathbb{Z}^d$.

We call a compact Abelian group G a *d-fold monothetic group* if G is the direct product of d monothetic groups $G_1, G_2, \dots, G_d$. Let g_j be a generator of G_j, for $j = 1, 2, \dots, d$, respectively. We refer to $g = (g_1, g_2, \dots, g_d)$ as a *generator* of G and write $(G, g) = \prod_{j=1}^d (G_j, g_j)$, where (G_j, g_j), $j = 1, 2, \dots, d$, are referred to as the *factor* monothetic groups of (G, g). Since the set

$$\big\{k \cdot g \equiv (k_1 g_1, k_2 g_2, \dots, k_d g_d) : k = (k_1, k_2, \dots, k_d) \in \mathbb{Z}^d\big\}$$

is dense in G, it is clear that $z \mapsto z + k \cdot g$ $(z \in G, k \in \mathbb{Z}^d)$ defines a $\mathbb{Z}^d$-action $(G, \mathbb{Z}^d)$ on G which we simply denote by (G, g). Moreover, the flows (G, g), (G, g') are conjugate for any two generators $g, g' \in G$.

The connection between d-fold monothetic groups and symbolic d-lattice dynamical systems is established by

Lemma 2.3. *Any minimal set of* $(\Sigma, \mathbb{Z}^d)$ *has a* d*-fold monothetic group as its maximal almost periodic factor.*

Proof. We note that the universal almost periodic minimal flow with $\mathbb{Z}^d$-action is the Bohr compactification of $\mathbb{Z}^d$, hence a d-fold monothetic group. The lemma immediately follows since any maximal almost periodic factor of a minimal set of $(\Sigma, \mathbb{Z}^d)$ is a factor of the universal almost periodic minimal flow. □

Let $M \subset \Sigma$ be an invariant set. We recall the definition of the topological entropy of M from [8]. Given any d-tuple $N = (N_1, N_2, \ldots, N_d) \in \mathbb{N}^d$, denote by

$$E_N := \big\{(k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d : 0 \le k_j \le N_j - 1 \text{ for } 1 \le j \le d\big\}$$

an $N_1 \times N_2 \times \cdots \times N_d$ parallelepiped in $\mathbb{Z}^d$. Let $\pi_{E_N} : \Sigma \to S^{E_N}$ be the natural projection and $\Gamma_N(M) = \operatorname{card} \pi_{E_N}(M)$. Then the *topological entropy* of M is defined as the limit

$$h(M) := \lim_{N \to \infty} \frac{1}{N_1 N_2 \cdots N_d} \log \Gamma_N(M)\,,$$

where $N \to \infty$ means that $N_j \to \infty$ for each $j = 1, 2, \ldots, d$. It has been shown in [8] that $h(M)$ is well defined and $0 \le h(M) \le \log m$.

3. Almost automorphic symbolic lattice dynamics

3.1. **Construction and characterization of almost automorphic symbolic dynamics.** In order to characterize almost automorphic symbolic lattice minimal flows, we generalize the concept of separating covers to d-fold monothetic groups. We thereby follow the work of [32] for the case $d = 1$.

Definition 3.1. A *separating cover* of a d-fold monothetic group (G, g) is an ordered finite cover $\alpha = \{D_0, D_1, \ldots, D_{m-1}\}$ of G satisfying

a) $\operatorname{cl}\big(\operatorname{int}(D_i)\big) = D_i$ for all $i = 0, 1, \ldots, m-1$;
b) $\operatorname{int}(D_i \cap D_j) = \emptyset$ for all $i \neq j$;
c) $D_i + z = D_i$ for $i = 0, 1, \ldots, m-1$ implies $z = 0$.

It is known that if (G, g) admits a separating cover then it must be metrizable ([31, 33]). Using a separating cover, almost automorphic arrays in the lattice system $(\Sigma, \mathbb{Z}^d)$ can be constructed as follows. Let $U = G \setminus \bigcup_i \partial D_i$ and $A = \bigcap_{k \in \mathbb{Z}^d} (U + k \cdot g)$. Then A is a residual subset of G. Fix a point $z \in A$. We define $x \in \Sigma$ to be such that for any $k \in \mathbb{Z}^d$,

$$x(k) = s_i, \quad \text{precisely if } \; z + k \cdot g \in D_i\,, \tag{3.1}$$

for $i = 0, 1, \ldots, m-1$.

Lemma 3.1. *Let* (G, g) *be a metrizable* d*-fold monothetic group having a finite cover* $\alpha = \{D_0, D_1, \ldots, D_{m-1}\}$.

(i) *If the properties* a),b) *in* Definition 3.1 *hold for* α, *then any array* x *constructed via* (3.1) *is almost automorphic.*
(ii) *If, in addition, property* c) *also holds, i.e., if* α *is a separating cover, then for the almost automorphic array* x *constructed via* (3.1), (G, g) *is a maximal almost periodic factor of the almost automorphic minimal set* $M = \operatorname{cl}\{x \cdot k : k \in \mathbb{Z}^d\} \subset \Sigma$.

Proof. (i) We note that

$$x(k) = \sum_{i=0}^{m-1} s_i \chi_{D_i}(z + k \cdot g), \quad k \in \mathbb{Z}^d ,$$

where χ_{D_i} denotes the characteristic function of D_i for $i = 0, 1, \dots, m-1$. For fixed i, we consider the function $F : \mathbb{Z}^d \to \{0, 1\}$ with $F(k) = \chi_{D_i}(z + k \cdot g)$. Let (k_n) be a sequence in $\mathbb{Z}^d$ such that $F(k + k_n) \to F_*(k)$ pointwise. Without loss of generality, we may assume that $k_n \cdot g$ converges in G. Then $(k_n - k_{n'}) \cdot g \to 0$ as $n, n' \to \infty$. Since, for any $k \in \mathbb{Z}^d$, $z + k \cdot g \subset \bigcup_{i=0}^{m-1} \mathrm{int} D_i$, we see that F is continuous at k. Hence $F(k - k_{n'} + k_n) \to F(k)$ as $n, n' \to \infty$. It now follows from the identity

$$F_*(k - k_n) - F(k) = \big(F_*(k - k_n) - F(k - k_n + k_{n'})\big) + \big(F(k - k_n + k_{n'}) - F(k)\big)$$

that F is almost automorphic, and so is the array x.

The proof of (ii) is completely analogous to the proof of Theorem 1.5 in [31] and Theorem 2.5 in [33]. We omit the details. □

The converse of the above lemma is also true.

Lemma 3.2. *Let $M \subset \Sigma$ be an almost automorphic minimal set with a d-fold monothetic group (G, g) as its maximal almost periodic factor (hence G is metrizable). Then (G, g) admits a separating cover from which any almost automorphic array $x \in M$ can be constructed via* (3.1).

Proof. Let $p : (M, \mathbb{Z}^d) \to (G, g)$ be the induced flow homomorphism and define $D_i := p(\{x \in M : x(0) = s_i\})$, $i = 0, 1, \dots, m-1$. Then it is easy to see that $\alpha = \{D_0, D_1, \dots, D_{m-1}\}$ is a separating cover of G, and any almost automorphic array $x \in M$ satisfies $x(k) = s_i$ if $p(x) + k \cdot g \in D_i$, for all $k \in \mathbb{Z}^d, i = 0, 1, \dots, m-1$. □

In general, it may be difficult to construct a separating cover with desired properties for a d-fold monothetic group. However, we do have

Lemma 3.3. *A d-fold monothetic group $(G, g) = \prod_{j=1}^{d} (G_j, g_j)$ admits a separating cover if each factor group (G_j, g_j) does.*

Proof. For each $j = 1, 2, \dots, d$ let $\alpha_j = \{D_0^j, D_1^j, \dots, D_{m-1}^j\}$ be a separating cover of (G_j, g_j). We obtain a finite cover

$$\big\{\mathcal{D}_n : n = (n_1, n_2, \dots, n_d) \in \{0, 1, \dots, m-1\}^d\big\}$$

by setting

$$\mathcal{D}_n := \prod_{j=1}^{d} D_{n_j}^j .$$

Let $S_d = \{0, 1, \dots, m-1\}^d$ and $f : S_d \to S$ be an onto map. It is easy to see that $\alpha = \{D_0, D_1, \dots, D_{m-1}\}$ with

$$D_i := \bigcup_{n \in S_d} \{\mathcal{D}_n : f(n) = s_i\}, \quad i = 0, 1, \dots, m-1 ,$$

is a separating cover of (G, g). □

Since any monothetic group admits a separating cover ([33]), the same holds for any d-fold monothetic group. It follows from Lemma 3.1 that given any d-fold monothetic group (G, g) there is an almost automorphic minimal subflow of $(\Sigma, \mathbb{Z}^d)$ having (G, g) as its maximal almost periodic factor.

Following a technique introduced in [32] we now construct separating covers of a compact metrizable d-fold monothetic group $(G, g) = \prod_{j=1}^{d}(G_j, g_j)$ from a so-called base set of separating covers.

Definition 3.2. Let (G, g) be a compact metrizable d-fold monothetic group. A *base set of separating covers* for (G, g) is a sequence $\{U_n\}_{n\in\mathbb{N}^d}$ of disjoint open sets of G with the following properties:

a) The set $C \subset G$ with
$$C := G \setminus \bigcup_{n\in\mathbb{N}^d} U_n\,,$$
is nowhere dense, and $C + z = C$ implies $z = 0$.

b) For any $z \in C$ and any neighborhood V of z, one has $V \cap U_n \neq \emptyset$ for infinitely many n.

Similarly to [32], one can use a base set $\{U_n\}_{n\in\mathbb{N}^d}$ of separating covers for (G, g) to construct a separating cover as follows. Let $\Sigma_+ = S^{\mathbb{N}^d}$ be the set of "one-sided" symbolic lattice arrays. Then each $\omega = \{\omega_n\}_{n\in\mathbb{N}^d} \in \Sigma_+$ defines the sets

$$D_i(\omega) := \mathrm{cl}\Big[\bigcup_{n\in\mathbb{N}^d}\{U_n : \omega_n = s_i\}\Big], \quad i = 0, 1, \ldots, m-1\,. \tag{3.2}$$

It is easy to see that $\alpha(\omega) = \{D_0(\omega), D_1(\omega), \ldots, D_{m-1}(\omega)\}$ is a finite cover of (G, g) satisfying the conditions a), b) in Definition 3.1. Hence by Lemma 3.1 (i) this cover generates an almost automorphic lattice array x_ω via (3.1), which in the present situation takes the form

$$x_\omega(k) = \omega_n, \quad \text{whenever } z + kg \in U_n\,. \tag{3.3}$$

In general, unless $\alpha(\omega)$ becomes a separating cover, one cannot be sure that the orbit closure of x_ω has (G, g) as its maximal almost periodic factor. The following lemma can be proved similarly to Lemma 2 in [32].

Lemma 3.4. *Given a base set of separating covers for (G, g) there exists a residual set $R \subset \Sigma_+$ such that for each $\omega \in R$ the cover $\alpha(\omega)$ according to (3.2) is a separating cover of (G, g) with $C = \partial\alpha(\omega)$.*

Base sets of separating covers for monothetic groups have been studied extensively in [32]. In particular, it is known that sets of separating covers exist for monothetic groups such as the p-adic odometer (adding machine associated to the group of p-adic integers) and the circle group. This allows us to construct a separating cover for a compact metrizable d-fold monothetic group $(G, g) = \prod_{j=1}^{d}(G_j, g_j)$ based on the base sets of separating covers for the factor monothetic groups, if they exist. For each $j = 1, 2, \ldots, d$, suppose that the group (G_j, g_j) admits a base set of separating covers $\{U_i^j\}_{i\in\mathbb{N}}$ and let $C^j = G_j \setminus \bigcup_{i\in\mathbb{N}} U_i^j$. Define

$$U_n := \prod_{j=1}^{d} U_{n_j}^j \tag{3.4}$$

for each $n = (n_1, n_2, \ldots, n_d) \in \mathbb{N}^d$. Then it is clear that $\{U_n\}_{n\in\mathbb{N}^d}$ satisfies both conditions a) and b) in Definition 3.2. Hence $\{U_n\}_{n\in\mathbb{N}^d}$ is a base of separating covers for (G, g), from which an almost automorphic lattice array x_ω, or equivalently, an almost automorphic minimal set $M(\omega) = \mathrm{cl}\{x_\omega \cdot k : k \in \mathbb{Z}^d\} \subset \Sigma$, can be obtained for each $\omega \in \Sigma_+$ by using (3.3). If (G, g) is metrizable, then according to Lemma 3.4 there are residually many ω in Σ_+ which generate separating covers of (G, g) such that $M(\omega)$ has (G, g) as its maximal almost periodic factor.

We now discuss the positivity of the topological entropy of the almost automorphic minimal set $M(\omega)$. Let $(G, g) = \prod_{j=1}^d (G_j, g_j)$ be metrizable and $\{U_i^j\}_{i\in\mathbb{N}}$ as above. For each $j \in \{1, 2, \ldots, d\}$ and each fixed $z_j \in \bigcap_{k\in\mathbb{Z}}(G_j \backslash C^j + kg_j)$, one can define inductively an increasing sequence of natural numbers $\{c_i^j\}_{i\in\mathbb{N}}$ such that $z_j + c_i^j g_j \in U_i^j$ and $z_j + lg_j \in \bigcup_{q=1}^{i-1} U_q^j$ for all $1 \le l < c_i^j$. (It may be necessary to re-index the sets U_i^j, see [32] for the details).

Theorem 3.1. *If $\underline{\lim}_{i\to\infty} c_i^j/i \le \delta < \infty$ for all $j = 1, \ldots, d$ and $m > 2^{\rho^d}$ with $\rho = \lfloor \delta \rfloor + 1$, then there exists a residual set $R \subset \Sigma_+$, depending on (G, g) and the above prescribed base set of separating covers $\{U_n\}$, such that for every $\omega \in R$ the almost automorphic minimal set $M(\omega)$ has (G, g) as its maximal almost periodic factor, is not uniquely ergodic, and has topological entropy $h\big(M(\omega)\big) \ge \rho^{-d} \log(2^{-\rho^d} m) > 0$.*

Proof. Define for $l \in \mathbb{N}$ and $j \in \{1, \ldots, d\}$ the set M_l^j as

$$M_l^j := \{i \in \mathbb{N} : c_{i+l}^j - c_{i+1}^j < \rho l\} .$$

The same argument as in [32] shows that M_l^j is infinite for all j, l. Consequently, $M_l := \prod_{j=1}^d M_l^j \subset \mathbb{N}^d$ is infinite, too. For any $(n_1, \ldots, n_d) \in M_l$ we have $c_{n_j+l}^j - c_{n_j+1}^j < \rho l$ for all $j = 1, \ldots, d$. Let $a \in S^{l^d}$ be any l^d-block built from the m symbols and set

$$Q_a := \big\{\omega \in \Sigma_+ : \{x_\omega(n+k)\}_{k\in\mathbb{N}^d, |k|\le l} \ne a \text{ for all } n \in M_l\big\} ,$$

where x_ω is determined according to (3.3) and $\{x_\omega(n+k)\}_{k\in\mathbb{N}^d,|k|\le l}$ denotes the l^d-block of x_ω having its “lower left corner” at $n + (1, \ldots, 1)$. Then Q_a is closed and nowhere dense in Σ_+, implying that

$$R_* := \bigcap_{l\in\mathbb{N}} \bigcap_{a\in S^{l^d}} Q_a^c \quad \subset \quad \Sigma_+$$

is residual. Now fix an $\omega \in R_*$. For any $l \in \mathbb{N}$ and any l^d-block a there exists $n \in M_l$ such that $\{x_\omega(n+k)\}_{k\in\mathbb{N}^d,|k|\le l} = a$. The point x_ω according to (3.3) therefore displays within $(\rho l)^d$-blocks all m^{l^d} possible l^d-blocks. The number of *different* $(\rho l)^d$-blocks in x_ω is thus at least

$$m^{l^d} \binom{(\rho l)^d}{l^d}^{-1} \ge m^{l^d} 2^{-(\rho l)^d} ,$$

which in turn implies that

$$h\big(M(\omega)\big) \ge \rho^{-d} \log(2^{-\rho^d} m) .$$

Similar to [32], we now claim that $M(\omega)$ cannot be uniquely ergodic. First of all, using the above argument, for any l, one can find a $(\rho l)^d$ block A_l in x_ω containing s_i at least l^d times. Next, consider for each symbol s_i the set $E_i = \{x \in M(\omega) :$

$x(0) = s_i\}$. If $M(\omega)$ is uniquely ergodic, then a straightforward application of the Birkhoff ergodic theorem to the continuous characteristic function $\chi_{E_i} : M(\omega) \to \mathbb{R}$ yields that the *frequency* of the symbol s_i

$$\gamma_i = \lim_{l\to\infty} \frac{\operatorname{card}\{n \in \mathbb{N}^d : x(n) = s_i, |n| \le l\}}{l^d} \tag{3.5}$$

is well defined, i.e., the limit exists uniformly and is independent of $x \in M(\omega)$. Given $\varepsilon > 0$, we have by the uniform convergence of (3.5) that there is an $L > 0$ such that

$$\frac{\operatorname{card}\{n \in \mathbb{N}^d : x_\omega(n+k) = s_i, |n| \le L\rho\}}{L^d} < \rho^d\gamma_i + \varepsilon, \text{ for all } k \in \mathbb{Z}^d. \tag{3.6}$$

In particular, let k be chosen such that $\{x_\omega(n+k)\}_{n\in\mathbb{N}^d, |n|\le L\rho} = A_L$. Then, with such a k the left hand side of (3.6) is greater or equal to 1. It follows that $\rho^d\gamma_i \ge 1$ for all $i = 0, 1, \cdots, m-1$, and hence

$$1 = \sum_{i=0}^{m-1} \gamma_i \ge \frac{m}{\rho^d} > 1,$$

a contradiction.

The proof is completed by taking the intersection of R_* with the residual set defined in Lemma 3.4. $\square$

We now consider the special case that $G = \mathbb{T}^d$, i.e. the d-torus. Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_d)$ be an irrational vector (that is, γ_j is irrational for every j) and define a $\mathbb{Z}^d$ flow $(\mathbb{T}^d, \gamma)$ as $z \mapsto z \cdot k = z + (\gamma_1 k_1, \gamma_2 k_2, \ldots, \gamma_d k_d)$ for all $z \in \mathbb{T}^d$, $k = (k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d$. Take Cantor sets $C^j \subset \mathbb{T}^1$ of positive Lebesgue measure $\lambda(C^j)$, $j = 1, 2, \ldots, d$, and write the complement of each Cantor set as a disjoint union of countably many intervals, i.e., $\mathbb{T}^1 \backslash C^j = \bigcup_{i=1}^\infty U_i^j$. It has been shown in [32] that for each j, $\{U_i^j\}_{i\in\mathbb{N}}$ is a base set of separating covers for $\mathbb{T}^1$, and moreover, fixing any $z^j \in A^j := \bigcap_{k\in\mathbb{Z}}(\{\mathbb{T}^1\backslash C^j\} + \gamma_j k)$, $j = 1, 2, \ldots, d$, one can define a sequence $\{c_i^j\}$ for which the condition of Theorem 3.1 is satisfied with $d = 1$ and $\delta = 2/\min_j \lambda(C^j)$. Let $\{U_n\}_{n\in\mathbb{N}^d}$ be a base set of separating covers for G generated from $\{U_i^j\}$ by making direct products as in (3.4). For each $\omega \in \Sigma_+$, let $M(\omega) = \operatorname{cl}\{x_\omega \cdot k : k \in \mathbb{Z}^d\}$ be the almost automorphic minimal set obtained from (3.3) using $\{U_n\}_{n\in\mathbb{N}^d}$. Then, by Theorem 3.1, there exists a residual set $R \subset \Sigma_+$ such that for any $\omega \in R$, $M(\omega)$ is not uniquely ergodic, of positive topological entropy, and has $(\mathbb{T}^d, \gamma)$ as its maximal almost periodic factor. In fact, more can be said about the topological entropy of such an almost automorphic minimal set.

Theorem 3.2. *Given $\eta \in (0, \log m)$. There exists a Cantor set $C \subset \mathbb{T}^1$, an irrational vector γ, and a residual set $R \subset \Sigma_+$, depending on η, C, γ, such that for every $\omega \in R$, the respective almost automorphic minimal set $M(\omega)$ has $(\mathbb{T}^d, \gamma)$ as its maximal almost periodic factor, is not uniquely ergodic and has topological entropy $h\big(M(\omega)\big) > \eta$.*

Proof. For simplicity, we only consider the case that $\gamma = (\gamma_0, \gamma_0, \cdots, \gamma_0)$. We start with a preliminary observation. Take $\gamma_0 \in \mathbb{R}$ and denote by $q_1 < q_2 < \ldots$ the sequence of denominators of the convergents to γ_0. We require that $\lim_{n\to\infty} q_{n+1}/q_n = +\infty$. (Though having measure zero the set of such γ_0 is dense on the real line.)

For D_n, i.e. the discrepancy of the sequence $(n\gamma_0)_{n\in\mathbb{N}}$, the following estimate is well known:

$$D_{q_n} \le q_n^{-1} + q_{n+1}^{-1} \quad (n \in \mathbb{N}) .$$

(We refer to [27] for details about continued fractions and uniform distribution.) Consequently, $\overline{\lim}_{n\to\infty} q_n D_{q_n} \le 1$.

Given now $\eta \in (0, \log m)$ fix a number $\kappa < 1$ sufficiently close to 1 such that $\kappa \log \kappa m + (1-\kappa)\log(1-\kappa) > \eta$, and let $C \subset \mathbb{T}^1$ be a Cantor set with $\lambda(C)^d > \kappa$. Setting $C^j := C$, $\mathbb{T}^1\backslash C = \bigcup_{i=1}^{\infty} U_i$, $U_i^j := U_i$ for $j = 1, \dots, d$ we may construct a base set of separating covers for $\mathbb{T}^d$ as described above. By the definition of discrepancy

$$\begin{aligned} q_n D_{q_n} &= \sup_{0\le\alpha<\beta\le1} \bigl|\mathrm{card}\{1 \le k \le q_n : \gamma_0 k + z_0 \in [\alpha, \beta)\} - q_n(\beta-\alpha)\bigr| \\ &\ge \bigl|\mathrm{card}\{1 \le k \le q_n : \gamma_0 k + z_0 \in U_i\} - q_n\lambda(U_i)\bigr| \end{aligned}$$

for all $n, i \in \mathbb{N}$. Therefore

$$q_n \le l_n q_n D_{q_n} + q_n\bigl(1 - \lambda(C)\bigr)$$

where $l_n := \max\{i : c_i \le q_n\}$ with the quantities $\{c_i\}$ as in Theorem 3.1. Consequently

$$\frac{c_{l_n}}{l_n} \le \frac{q_n}{l_n} \le \lambda(C)^{-1} q_n D_{q_n} \le \lambda(C)^{-1}\bigl(1 + \frac{q_n}{q_{n+1}}\bigr) .$$

Since $l_n \to \infty$ as $n \to \infty$, $\underline{\lim}_{n\to\infty} c_n/n \le \lambda(C)^{-1}$. By construction $c_i^j = c_i$, and therefore Theorem 3.1 applies with $\delta = \lambda(C)^{-1}$. A careful re-examination of the proof of that theorem shows that the argument remains valid for every $\rho > \delta$ (see also Theorem 3.4 below). Now, for any ω from the appropriate residual set $R \subset \Sigma_+$,

$$\begin{aligned} h\bigl(M(\omega)\bigr) &\ge \overline{\lim}_{l\to\infty} \lfloor \rho l \rfloor^{-d} \log\Bigl(m^{l^d} \binom{\lfloor \rho l\rfloor^d}{l^d}^{-1}\Bigr) \\ &= \rho^{-d}\log m - d\log\rho + (1-\rho^{-d})\log(\rho^d - 1) \end{aligned}$$

by virtue of Stirling's formula. Taking ρ sufficiently close to $\delta = \lambda(C)^{-1}$ we thus have

$$h\bigl(M(\omega)\bigr) \ge \kappa \log m + \log\kappa + (1-\kappa)\log(\kappa^{-1} - 1) > \eta .$$

An application of Lemma 3.4 and Theorem 3.1 (providing residual sets for the other claimed properties) therefore completes the proof. □

3.2. **Toeplitz arrays.** Toeplitz sequences form an important class of almost automorphic points in the usual symbolic dynamical system $(\Sigma, \mathbb{Z}^1)$. In fact, it is known that a minimal set of $(\Sigma, \mathbb{Z}^1)$ is Toeplitz (i.e., closure of a Toeplitz sequence) if and only if it is almost automorphic with a 0-dimensional (totally disconnected) monothetic group (odometer, adding machine) as the maximal almost periodic factor (see [10, 32]). We now define Toeplitz arrays in the lattice case.

Definition 3.3. A lattice array $\{x(k)\} \in \Sigma$ is said to be *Toeplitz* if for any $k \in \mathbb{Z}^d$ there is $n = (n_1, n_2, \dots, n_d) \in \mathbb{N}^d$ such that $x(k) = x(k')$ for all $k' \in \mathbb{Z}^d$, $k' \equiv k$ (mod n). A *Toeplitz set* is the orbit closure of a Toeplitz array.

Theorem 3.3. *A minimal set of $(\Sigma, \mathbb{Z}^d)$ is Toeplitz if and only if it is almost automorphic with a* 0*-dimensional d-fold monothetic group as its maximal almost periodic factor. Moreover, the Toeplitz arrays in such a minimal set are precisely the almost automorphic points.*

Proof. Let $(E, \mathbb{Z}^d)$ be an almost automorphic minimal subflow of $(\Sigma, \mathbb{Z}^d)$ with a 0-dimensional d-fold monothetic group (G, g) as its maximal almost periodic factor. We denote by $p : (E, \mathbb{Z}^d) \to (G, g)$ the induced flow homomorphism. Let E_0 be the set of almost automorphic points in E. Then by Lemma 2.1 $A = p(E_0)$ is a residual subset of G and $p^{-1}p(x) = \{x\}$ for all $x \in E_0$. Define $f : A \to S$ as $f\big(p(x)\big) := x(0)$ for all $x \in E_0$. Since E_0 consists of singleton fibers and A is invariant (i.e., $A + kg \subset A$ for all $k \in \mathbb{Z}^d$), f is continuous and $f(p(x) + kg) = p^{-1}(p(x) + k \cdot g)(0) = p^{-1}p(x)(k) = x(k)$, for all $x \in E_0$ and $k \in \mathbb{Z}^d$.

We now fix an $x \in E_0$ and $k \in \mathbb{Z}^d$. Let $x(k) = f(p(x) + k \cdot g) = s_i \in S$. By continuity, there exists an open neighborhood U_k^* of $p(x) + k \cdot g$ such that $f(z') = s_i$ whenever $z' \in U_k^* \cap A$. Let $U_k = U_k^* - p(x) - k \cdot g$. Then U_k is a neighborhood of $0 \in G$. Hence there is a nontrivial open subgroup $H \subset U_k$ ([18]), and $f(p(x) + k \cdot g + h) = s_i$ whenever $h \in H$ and $z + k \cdot g + h \in A$. Let $H_k = \{n : n \cdot g \in H\}$. It is clear that H_k is a subgroup of $\mathbb{Z}^d$, and in fact $H_k = \prod_{j=1}^d n_j \mathbb{Z}$, for some $n_j \in \mathbb{N}$, $j = 1, 2, \ldots, d$. Let $n_0 = (n_1, n_2, \ldots, n_d)$ and assume that $k' \equiv k \pmod{n_0}$. Then $k' = k + k''$ for some $k'' \in H_k$ and $p(x) + k'' \cdot g \in U_k^* \cap A$. Hence $x(k') = f(p(x) + k' \cdot g) = s_i = x(k)$. This shows that x is a Toeplitz array.

The converse is a straightforward generalization of a result in [43]. Let $x \in \Sigma$ be a Toeplitz array. For each $n = (n_1, n_2, \ldots, n_d) \in \mathbb{N}^d$ we consider the set

$$Per_n = \bigcup_{s_i \in S} Per_n(s_i)\,,$$

where

$$Per_n(s_i) := \{k \in \mathbb{Z}^d : x(k') = s_i \text{ for all } k' \equiv k \,(\mathrm{mod}\, n)\}.$$

It is clear that $Per_n \subset Per_{n'}$ whenever $n | n'$ (i.e., each component of n divides the corresponding component of n'). Since x is Toeplitz, we have

$$\mathbb{Z}^d = \bigcup_{n \in \mathbb{N}^d} Per_n\,. \tag{3.7}$$

Consider the set P of all $n \in \mathbb{N}^d$ such that whenever $Per_n(s_i) = Per_n(s_i) + n'$ for all $s_i \in S$ then $n|n'$. For $n, n' \in P$ let n^* be the d-tuple each component of which equals the least common multiple of the corresponding components in n, n'. It is easy to see that $n^* \in P$. This implies that P is a directed set. The quotient groups

$$G_n = \prod_{j=1}^d \left(\frac{\mathbb{Z}}{n_j \mathbb{Z}} \right), \quad n = (n_1, n_2, \ldots, n_d) \in P\,,$$

form an inverse system and therefore admit an inverse limit G which is easily seen to be a direct product of d odometer groups. Hence G is a 0-dimensional d-fold monothetic group. Due to (3.7) we also have

$$\mathbb{Z}^d = \bigcup_{n \in P} Per_n\,.$$

Using this fact and an argument analogous to [43] one can readily show that G is a maximal almost periodic factor of the orbit closure of x. $\square$

In [32] an important class of Toeplitz sequences is constructed using groups of p-adic integers. We now generalize this construction to the lattice case with particular attention on Toeplitz lattice arrays with large topological entropy. Fix a family of prime numbers p_j, $j = 1, 2, \dots, d$, and for $i \in \mathbb{N}$, $j = 1, 2, \dots, d$ let

$$U_i^j := (c_i^j + p_j^{N^j+i}\mathbb{Z}) \cup (-c_i^j + p_j^{N^j+i}\mathbb{Z}) \subset \mathbb{Z},$$

where $N^j \in \mathbb{N}$ will be chosen later. (If some $p_j = 2$ we correspondingly require $N^j \geq 2$ in order to obtain infinitely many different sets). The integers $c_i^j \geq 0$ are determined inductively according to

$$c_1^j := 0 \quad \text{and} \quad c_{i+1}^j := \min\{n \geq 0 : n \notin \bigcup_{l=1}^{i} U_l^j\}.$$

For each j the disjoint union of the sets U_i^j equals $\mathbb{Z}$, and $C^j = \Delta_{p_j} \setminus \bigcup_{i=1}^{\infty} U_i^j$ is nowhere dense, where $(\Delta_{p_j}, \oplus_1)$ denotes the p_j-adic odometer group (adding machine) associated to the group of p_j-adic integers [26]. It has been shown in [32] that the sets $C^j, \{U_i^j\}$ together with the sequence $\{c_i^j\}$ satisfy the conditions of Theorem 3.1 with $d = 1$. Again, let $\{U_n\}_{n \in \mathbb{N}^d}$ be the base of separating covers for G generated from $\{U_i^j\}_{i \in \mathbb{N}, 1 \leq j \leq d}$ via the direct product (3.4). For each $\omega \in \Sigma_+$, we let $M(\omega) = \mathrm{cl}\{x_\omega \cdot k : k \in \mathbb{Z}^d\}$ be the almost automorphic minimal set obtained from (3.3) using $\{U_n\}_{n \in \mathbb{N}^d}$. That is, for each $\omega = \{\omega_n\} \in \Sigma_+$, $x_\omega \in \Sigma$ is a Toeplitz array defined by setting $x_\omega(k) := \omega_n$ whenever $k \in U_n$.

Theorem 3.4. *Given $\eta \in (0, \log m)$ there exists a residual set $R \subset \Sigma_+$, depending on $\eta, p_1, p_2, \cdots, p_d$, such that for every $\omega \in R$, the almost automorphic minimal set $M(\omega)$ has the group $\prod_{j=1}^d (\Delta_{p_j}, \oplus_1)$ as its maximal almost periodic factor, is not uniquely ergodic and has topological entropy $h\big(M(\omega)\big) > \eta$.*

Proof. The maximal (topological) entropy of $(\Sigma, \mathbb{Z}^d)$ amounts to $\log m$. We shall now demonstrate that the above construction typically yields an almost automorphic system with entropy close to this maximal value. The argument is a refinement of the proof of Theorem 3.1. Take $0 < \varepsilon < \frac{1}{2}$. We let N^j be such that $\sum_{i \in \mathbb{N}} p_j^{-(N^j+i)} < \varepsilon/4$. Then

$$[-c_i^j + 1, c_i^j - 1] \cap \mathbb{Z} \subset \bigcup_{l=0}^{i-1} U_i^j$$

and

$$\mathrm{card}(U_i^j \cap [-c_i^j + 1, c_i^j - 1]) \leq 2p_j^{-(N^j+i)}(2c_i^j - 1) + 2$$

so that $\underline{\lim}_{i \to \infty} c_i^j / i < 1 + \varepsilon$ for all j. For any $l \in \mathbb{N}$ the set $M_l^j := \{i \in \mathbb{N} : c_{i+l}^j - c_{i+1}^j < (1+\varepsilon)l\}$ therefore is infinite (cf. the proof of Theorem 3.1). Define $M_l = \prod_{j=1}^d M_l^j$ and $A_l = \{1, 2, \dots, l\}^d$. Then, given $\omega \in \Sigma_+$, we see any l^d-block of the m symbols infinitely often within blocks of size $\lfloor (1+\varepsilon)l \rfloor^d$ in Σ_+. Intuitively, it is clear that ω may be chosen in such a way that it contains *all possible* l^d-blocks at the right place. More precisely, proceed as in the proof of Theorem 3.1 by drawing an l^d-block a from S, i.e. $a \in S^{l^d}$ and let

$$Q_a := \{\omega \in \Sigma_+ : \{x_\omega(n+k)\}_{k\in\mathbb{N}^d, |k|\le l} \neq a \text{ for all } (n_1, \dots, n_d) \in M_l\} .$$

Clearly, Q_a is a closed, nowhere dense set, and hence

$$\widetilde{R} := \bigcap_{l\in\mathbb{N}} \bigcap_{a\in S^{l^d}} Q_a^c \quad\subset\quad \Sigma_+$$

is residual. Given $\omega \in \widetilde{R}$ and any l^d-block a one may thus find an index $n = (n_1, n_2, \dots, n_d) \in \mathbb{N}^d$ such that $\{x_\omega(n+k)\}_{k\in\mathbb{N}^d,|k|\le l} = a$ and $c^j_{n_j+l} - c^j_{n_j+1} < (1+\varepsilon)l$ for all $j = 1, 2, \dots, d$. Consequently, x_ω with $\omega \in \widetilde{R}$ contains all m^{l^d} possible l^d-blocks within blocks whose sizes are at most $\lfloor(1+\varepsilon)l\rfloor^d$. We can therefore find at least

$$m^{l^d} \begin{pmatrix} \lfloor(1+\varepsilon)l\rfloor^d \\ l^d \end{pmatrix}^{-1}$$

different blocks of size $\lfloor(1+\varepsilon)l\rfloor^d$ within x_ω. A straightforward application of Stirling's formula yields

$$\lim_{l\to\infty}\Big(\lfloor(1+\varepsilon)l\rfloor^{-d}\log\big(m^{l^d}\begin{pmatrix} \lfloor(1+\varepsilon)l\rfloor^d \\ l^d \end{pmatrix}^{-1}\big)\Big) \ge$$
$$(1+\varepsilon)^{-d}\log m - d\log(1+\varepsilon) + \big(1-(1+\varepsilon)^{-d}\big)\log\big((1+\varepsilon)^d - 1\big) .$$

Choosing ε appropriately and taking $\omega \in R \cap \widetilde{R}$, where R is as in Lemma 3.4, we have therefore proved the theorem. □

We remark that the residual sets in Theorems 3.1, 3.2, and 3.4 above have full measure with respect to any Bernoulli measure giving positive weight to every symbol. Therefore, the generic properties listed in these theorems are also typical from a statistical point of view.

Corollary 3.1. *Given prime numbers $p_1, p_2, \cdots, p_d$, there exists a residual set $R \subset \Sigma_+$, depending on $p_1, p_2, \cdots, p_d$, such that for every $\omega \in R$, the almost automorphic minimal set $M(\omega)$ has the group $\prod_{j=1}^d (\Delta_{p_j}, \oplus_1)$ as its maximal almost periodic factor, is not uniquely ergodic and has topological entropy $h\big(M(\omega)\big) = \log m$.*

Proof. Let $\eta_n \to \log m$ be an increasing sequence and $R_n \subset \Sigma_+$ be the residual set associate to each η_n according to Theorem 3.4. Then $R = \bigcap_{n=1}^\infty R_n$ clearly satisfies the properties stated in the corollary. □

4. Almost automorphic dynamics, pattern formation and spatial chaos

A (continuous-time) lattice dynamical system is an infinite system of ODEs, coordinated by a lattice $\mathbb{Z}^d$, of the form

$$\frac{du_l}{dt} = F_n(\{u_k\}_{k\in\mathbb{Z}^d}), \quad l \in \mathbb{Z}^d ,$$

which naturally generates a bi-transformation dynamical system with time dynamics defined by the time evolution and spatial dynamics defined by the group of shift maps $\sigma^l u_k = u_{l+k}$, $l, k \in \mathbb{Z}^d$.

As a special case, let us exam the discrete Allen-Cahn equations originally considered in [8] as models for pattern formation and spatial chaos in lattice systems. The models in 1D and 2D have the form

$$\dot{u}_l = -\beta\Delta u_l - f(u_l), \quad l \in \mathbb{Z}\,, \tag{4.1}$$

and

$$\dot{u}_{l,k} = -\beta^+\Delta^+ u_{l,k} - \beta^\times\Delta^\times u_{l,k} - f(u_{l,k}), \quad (l,k) \in \mathbb{Z}^2\,, \tag{4.2}$$

respectively, where Δ is the usual discrete Laplace operator in $\mathbb{Z}$, and Δ^+ and $\Delta^\times$ are two discrete Laplace operators, given respectively by the nearest neighbor and the next nearest neighbor stencils on the lattice $\mathbb{Z}^2$ ([8]); also, f is the piecewise linear function defined by

$$f(u) = f^\epsilon(u) = \begin{cases} \epsilon^{-1}(u+1) - \alpha & \text{if } u \le -1\,, \\ \alpha u & \text{if } |u| \le 1\,, \\ \epsilon^{-1}(u-1) + \alpha & \text{if } u \ge 1\,. \end{cases} \tag{4.3}$$

Taking $\epsilon \to 0$ in (4.3), we obtain the set valued function

$$f^0(u) = \begin{cases} (-\infty, -\alpha] & \text{if } u = -1\,, \\ \alpha u & \text{if } |u| < 1\,, \\ [\alpha, \infty) & \text{if } u = 1\,, \\ \emptyset & \text{if } |u| > 1\,. \end{cases} \tag{4.4}$$

Hence with the limiting function f^0, (4.1), (4.2) become differential inclusions, and the solutions u_l, $u_{l,k}$ can only take values in the interval $[-1,1]$.

Since the discrete Allen-Cahn equations are gradient systems (with respect to a reasonable norm on the phase space [8]), the structure of equilibrium solutions of these equations is vital for the analysis. The analysis is of particular interests when $\beta, \beta^+, \beta^\times$ are allowed to change signs, because of the rich dynamics which (4.1), (4.2) correspond. Equilibrium solutions of the limiting equation of (4.1), (4.2) which only assume values in $\{-1,0,1\}$ are called *mosaic solutions*. (All sequences in $\{-1,0,1\}^{\mathbb{Z}}$ and arrays in $\{-1,0,1\}^{\mathbb{Z}^2}$ are referred to as *mosaics*.) As in [8], two special sets of attracting mosaic solutions (called $\mathcal{S}$-*solutions*), denoted by $\mathcal{S}_1 = \mathcal{S}_1(\beta,\alpha), \mathcal{S}_2 = \mathcal{S}_2(\beta^+,\beta^\times,\alpha)$ for (4.1), (4.2) respectively, are of particular interest in the sense that (4.1) or (4.2) is said to exhibit *spatial chaos* (*pattern formation*, respectively) for the parameter value (β,α) or $(\beta^+,\beta^\times,\alpha)$, if the topological entropy of $\mathcal{S}_1(\beta,\alpha)$ or $\mathcal{S}_2(\beta^+,\beta^\times,\alpha)$ is positive (zero respectively). A complete classification of parameter values corresponding to either pattern formation or spatial chaos has been provided in [8] for both dimensions. It turns out that a large class of regular patterns discussed in [8] are actually almost automorphic, in fact, Toeplitz.

We now discuss spatial chaos responding to the onset of almost automorphic dynamics. From the definition of the discrete Laplacians in (4.1), (4.2), it is clear that each component of a mosaic solution is constrained by (or coupled to) its nearest neighbors (and the next nearest neighbors in the 2D case). In fact, as shown in [8], for each parameter value $(\beta^+,\beta^\times,\alpha)$ ((β,α), resp.), the set $\mathcal{S}_2(\beta^+,\beta^\times,\alpha)$ ($\mathcal{S}_1(\beta,\alpha)$, resp.) consists of arrays built from a set $\mathcal{S}$ of *admissible blocks* (*words*) of a fixed size less than or equal to three. More precisely, depending on the value of parameters, there are integers $1 \le A, B \le 3$ (one integer $1 \le A \le 3$, resp.) and a set $\mathcal{S}$ of finitely many $A \times B$ blocks (word of length A) such that for each $\{u_{l,k}\} \in \mathcal{S}_2$ ($\{u_l\} \in \mathcal{S}_1$) one has $\{u_{l+a,k+b}\}_{1\le a\le A, 1\le b\le B} \in \mathcal{S}$ ($\{u_{l+a}\}_{1\le a\le A} \in \mathcal{S}$, resp.) for all

$(l, k) \in \mathbb{Z}^2$ ($l \in \mathbb{Z}$). In general, admissible blocks (words) do not appear in an array of $\mathcal{S}_2$ (sequence of $\mathcal{S}_1$) in arbitrary order. Nevertheless, it is proved in [8] that $\mathcal{S}_{1,2}$ are equivalent to Markov subshifts of $(\mathcal{S}^{\mathbb{Z}^d}, \mathbb{Z}^d)$, $d = 1, 2$.

To explore the construction of a certain class of arrays via admissible blocks, let us consider the general situation $(\Sigma, \mathbb{Z}^d)$ as in the previous section. Given positive integers M, $N = \{N_1, N_2, \ldots, N_d\}$, we let $\mathcal{S} = \{S_1, S_2, \ldots, S_M\}$ be a set of $N_1 \times N_2 \times \cdots \times N_d$ *building blocks* in $(\Sigma, \mathbb{Z}^d)$. We treat $\mathcal{S}$ as a new alphabet set and consider the new symbolic d-lattice space $\Xi = \mathcal{S}^{\mathbb{Z}^d}$ carrying the $\mathbb{Z}^d$-shift action defined earlier.

Let $\mathcal{U} \subset \Sigma$ be the set of arrays built from $\mathcal{S}$, i.e., any array in $\mathcal{U}$ is obtained from "opening up" the blocks of an array in Ξ. Hence there is an one to one correspondence between Ξ and $\mathcal{U}$. Moreover, if we let $\mathbb{Z}_N = \prod_{j=1}^d N_j\mathbb{Z}$, then it is clear that $(\Xi, \mathbb{Z}^d)$ is topologically conjugate to the subshift $(\mathcal{U}, \mathbb{Z}_N)$.

Lemma 4.1. $h(\Xi) = \log M = (N_1 N_2 \ldots N_d) h(\mathcal{U})$.

Proof. It is easy to see that the entropy of $(\mathcal{U}, \mathbb{Z}_N)$ equals $(N_1 N_2 \cdots N_d) h(\mathcal{U})$. The lemma then follows easily from the conjugacy between $(\Xi, \mathbb{Z}^d)$ and $(\mathcal{U}, \mathbb{Z}_N)$. $\square$

We now take specifically $(G, g) = (\mathbb{T}^d, \gamma)$ or $\prod_{j=1}^d (\Delta_{p_j}, \oplus_1)$ with a base set of separating covers $\{U_n\}_{n \in \mathbb{N}^d}$ as in Theorem 3.2 or Theorem 3.4, respectively. Consider the set $\Xi_+ = \mathcal{S}^{\mathbb{N}^d}$. As before, there is a residual set $R \subset \Xi_+$ such that each $\omega \in R$ defines an almost automorphic minimal set in $(\Xi, \mathbb{Z}^d)$ and hence in $(\mathcal{U}, \mathbb{Z}_N)$ which corresponds to an almost automorphic minimal set $M(\omega)$ in $(\mathcal{U}, \mathbb{Z}^d)$ by Lemma 2.2. As an immediate consequence of the above lemma and Theorem 3.2 or Theorem 3.4 we have

Corollary 4.1.

(i) *For any $\eta \in \big(0, h(\mathcal{U})\big)$, there is an irrational vector $\gamma \in \mathbb{R}^d$ and a residual set $R \subset \Xi_+$ such that for each $\omega \in R$, $M(\omega)$ has $(\mathbb{T}^d, \gamma)$ as its maximal almost periodic factor, is not uniquely ergodic and has topological entropy $h\big(M(\omega)\big) > \eta$.*

(ii) *For any prime numbers $p_1, p_2, \cdots, p_d$, there is a residual set $R \subset \Xi_+$ such that for each $\omega \in R$, $M(\omega)$ has $\prod_{j=1}^d (\Delta_{p_j}, \oplus_1)$ as its maximal almost periodic factor, is not uniquely ergodic and attains the maximal topological entropy $h\big(M(\omega)\big) = h(\mathcal{U})$.*

We note that $M(\omega)$ is non-Toeplitz in the case (i) and Toeplitz in the case (ii). Thus, almost automorphic minimal sets of $(\mathcal{U}, \mathbb{Z}^d)$ which are non-uniquely ergodic with nearly maximal topological entropy, can have completely different topological and harmonic structure. (Recall that the dual group of the maximal almost periodic factor of an almost automorphic minimal set $M(\omega)$ defines its frequency module.)

Going back to the lattice models (4.1),(4.2), we consider the case that mosaics only take values in $\{-1, 1\}$. It is shown in [8] that there are various cases in which $\mathcal{S}$-solutions can be built from available building blocks as above. For instance, in the 1D case with $\alpha < 0$, $\mathcal{S}_1 = \{-1, 1\}^{\mathbb{Z}}$, and in the 2D case the set $\mathcal{S}_2$ for each case considered in Theorem 6.1 of [8], contain a subset $\mathcal{W}$ of $\mathcal{S}$-solutions which are built from either two 2×1 or two 2×2 admissible building blocks. Applying Corollary 4.1 we conclude that non-Toeplitz chaotic almost automorphic dynamics form a large subset of $\mathcal{W}$ with topological entropies arbitrarily close to that of $\mathcal{W}$, and Toeplitz chaotic almost automorphic dynamics actually form a large subset of $\mathcal{W}$ with the maximal topological entropy of $\mathcal{W}$. This gives strong evidence that chaotic almost automorphic dynamics should largely be responsible for the spatial chaos occurring in lattice models like (4.1), (4.2).

5. Construction of chaotic almost automorphic signals

In this final section, we propose a new approach to the construction of chaotic almost automorphic signals based on the study in the previous sections as well as on interpolation and suspension techniques. By a *chaotic signal* we mean a function f whose topological hull $H(f)$ as a (time-)translation dynamical system has positive topological entropy.

Let $(V, \|\cdot\|)$ denote a finite-dimensional normed space and consider $X := V^{\mathbb{Z}}$ endowed with the product topology. To be specific we shall think of this topology as being induced by the metric

$$d(v,w) \;=\; d\big((v_k)_{k\in\mathbb{Z}},(w_k)_{k\in\mathbb{Z}}\big) \;:=\; \sum_{k\in\mathbb{Z}} 2^{-|k|}\frac{\|v_k-w_k\|}{1+\|v_k-w_k\|}\,.$$

By $X_\infty \subset X$ we denominate the subset of *bounded* sequences in V. Dynamics on X originate from the continuous (left) shift map $\sigma : X \to X$ with $\sigma(v)_k := v_{k+1}$ for all k. Furthermore, let $C(\mathbb{R},V)$ stand for the space of continuous functions from $\mathbb{R}$ to V, endowed with the compact-open topology. Analogously, there is a continuous shift dynamics on $C(\mathbb{R},V)$ given by $\theta_t f := f(\cdot + t)$ for all $t \in \mathbb{R}$. If f is uniformly continuous (e.g., f is almost automorphic under θ), then the hull $H(f) = \overline{O_\theta(f)} = \overline{\{\theta_t f : t \in \mathbb{R}\}}$ is compact.

We now fix $e \in V$ and define a uniformly continuous map $\Phi_e : X \to C(\mathbb{R},V)$ by setting

$$\Phi_e v(s) := \sum_{k\in\mathbb{Z}}\big(v_{\lfloor k/2\rfloor}\max\{0,1-2|2s-k|\} + e\max\{0,1-2|2s-2k-1|\}\big)$$

for all $v = (v_k)_{k\in\mathbb{Z}} \in X$. Graphically, the function $\Phi_e v$ linearly interpolates the points (k, v_k), $(k+\frac14, 0)$, $(k+\frac12, v_k+e)$ and $(k+\frac34, 0)$ for all k (see Figure 5.1). More smoothness could have been assigned to the function $\Phi_e v$, but the simple Lipschitz version chosen here seems to be sufficient for potential applications. Obviously, $\Phi_e \circ \sigma = \theta_1 \circ \Phi_e$, and with $\Psi : C(\mathbb{R},V) \to X$ according to $\Psi(f) := \big(f(k)\big)_{k\in\mathbb{Z}}$ we have $\Psi \circ \Phi_e = id_X$.

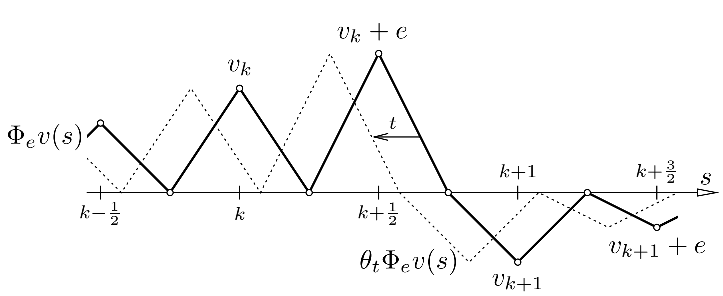

Figure 5.1. The continuous function $\Phi_e v$ interpolates the sequence $(v_k)_{k\in\mathbb{Z}}$.

Fixing $v \in X_\infty$ we write $X(v) := \overline{O_\sigma(v)}$ and $H(v) := \overline{O_\theta(\Phi_e v)}$ for the respective compact orbit closures. For the sake of brevity the restrictions of σ and θ to $X(v)$ and $H(v)$, respectively, will again be denoted by σ and θ.

Theorem 5.1. *Let $e \neq 0$ and $v \in X_\infty$. Then the suspension flow $(X(v)^\sigma, \varphi^\sigma)$ of $(X(v), \sigma)$ is topologically conjugate to $(H(v), \theta)$. Consequently, $h\big(X(v)\big) = h\big(H(v)\big)$, and $H(v)$ is almost automorphic (almost periodic, periodic, uniquely ergodic) with respect to θ if and only if $X(v)$ has the corresponding property with respect to σ.*

Proof. We first notice that σ is a homeomorphism of the compact space $X(v)$, and clearly $\Phi_e\big(X(v)\big) \subset H(v)$. Define now a map $\alpha : X(v)^\sigma \to H(v)$ by setting $\alpha(a, w) := \theta_a(\Phi_e w)$ for all $(a, w) \in X(v)^\sigma$. Evidently, α is continuous, and by virtue of $\Phi_e \circ \sigma = \theta_1 \circ \Phi_e$ also $\alpha \circ \varphi_t^\sigma = \theta_t \circ \alpha$ for all $t \in \mathbb{R}$. We claim that every $f \in H(v)$ has the form $\alpha(a, w)$ for some $(a, w) \in X(v)^\sigma$. Indeed, whenever $\theta_{t_n}(\Phi_e v) = \theta_{t_n - \lfloor t_n \rfloor}\big(\Phi_e \circ \sigma^{\lfloor t_n \rfloor}(v)\big) \to f$ we may assume that (for some subsequence whose subscripts we suppress) $t_n - \lfloor t_n \rfloor \to a' \in [0, 1]$ and $\sigma^{\lfloor t_n \rfloor}(v) \to w' \in X(v)$. By continuity we have $f = \alpha(a, w)$ with $a = a' - \lfloor a' \rfloor$ and $w = \sigma^{\lfloor a' \rfloor}(w')$. We may thus assign to each $f \in H(v)$ an element $(a, w) \in X(v)^\sigma$ which satisfies $f = \alpha(a, w)$. It remains to show that this assignment is well-defined, for then α is one-to-one and thus a homeomorphism. So let $\alpha(a, w) = \alpha(b, x)$, that is, $\theta_a(\Phi_e w) = \theta_b(\Phi_e x)$, or equivalently, $\theta_{a-b}(\Phi_e w) = \Phi_e x$. Since $e \neq 0$ we must have $a - b \in \mathbb{Z}$, and even $a = b$ as $0 \leq a, b < 1$. But Φ_e is one-to-one, implying that $(a, w) = (b, x)$. The remaining assertions of the theorem follow from Proposition 2.1. $\square$

We close this section by briefly discussing three examples.

Example 5.1. (*Non-chaotic almost automorphic signals with two frequencies*) We take $V = \mathbb{R}$ and fix an irrational number ϑ. It is easy to see that the sequence $v = (\cos 2\pi\vartheta k)_{k \in \mathbb{Z}} \in X$ defines an almost periodic point w.r.t. σ. The above construction thus yields an almost periodic function $\Phi_e v : \mathbb{R} \to \mathbb{R}$. In this case, the assignment $\sigma^l v \mapsto e^{2\pi i \vartheta l}$ $(l \in \mathbb{Z})$ may straightforwardly be extended to yield a topological conjugacy between $(X(v), \sigma)$ and the rotation R_ϑ of the unit circle $\mathbb{T}^1$ by the angle $2\pi\vartheta$. Hence $(H(v), \theta)$ is flow isomorphic to the minimal Kronecker flow $(R_{(t,\vartheta t)})_{t \in \mathbb{R}}$ of the two-torus $\mathbb{T}^2 = \mathbb{T}^1 \times \mathbb{T}^1$; here $R_{(\vartheta_1, \vartheta_2)} := R_{\vartheta_1} \times R_{\vartheta_2}$ for all $\vartheta_1, \vartheta_2 \in \mathbb{R}$. Consider now the point $v' := \big(\mathrm{sign}(\cos 2\pi\vartheta k)\big)_{k \in \mathbb{Z}} \in X$ where, as usual, $\mathrm{sign}(r)$ equals $+1$, 0, or -1, depending on whether $r > 0, r = 0$ or $r < 0$, respectively. An elementary calculation confirms that v' is almost automorphic but not almost periodic w.r.t. σ ([41]). Still the assignment $\sigma^l v' \mapsto e^{2\pi i \vartheta l}$ $(l \in \mathbb{Z})$ gives rise to a continuous map $p : X(v') \to \mathbb{T}^1$ which satisfies $p \circ \sigma = R_\vartheta \circ p$. However, if $z \in \{\pm i e^{2\pi i \vartheta k} : k \in \mathbb{Z}\} = O_{R_\vartheta}(i) \cup O_{R_\vartheta}(-i)$ then there are two p-pre-images to z in $X(v')$. For $e \neq 0$ therefore $f = \Phi_e v'$ is an almost automorphic signal whose orbit closure has the above Kronecker flow as its almost periodic factor. Moreover, $H(v')$ may be thought of as $\mathbb{T}^2$ with two trajectories "doubled". Since $(\mathbb{T}^1, R_\vartheta)$ is uniquely ergodic with respect to the (normalized) Haar measure $\lambda_{\mathbb{T}^1}$ and $\lambda_{\mathbb{T}^1}\big(O_{R_\vartheta}(i) \cup O_{R_\vartheta}(-i)\big) = 0$, it follows that $(H(v'), \theta)$ is also uniquely ergodic with respect to (the appropriately lifted version of) $\lambda_{\mathbb{T}^2} = \lambda_{\mathbb{T}^1} \otimes \lambda_{\mathbb{T}^1}$, and $h(\theta) = 0$. Thus, f is non-chaotic with frequency module $\mathcal{M}(f) \sim \{n + \vartheta m : n, m \in \mathbb{Z}\}$ and so admits two (basic) frequencies $\{1, \vartheta\}$.

Example 5.2. (*Chaotic almost automorphic signals with two frequencies*) We consider $V = \mathbb{R}$ and the full shift $(\Sigma, \mathbb{Z}) = (S^{\mathbb{Z}}, \sigma)$ on the m-symbol alphabet $S = \{s_0, s_1, \dots, s_{m-1}\}$. Note that Σ can homeomorphically be embedded into $X = \mathbb{R}^{\mathbb{Z}}$ via $i_m : \Sigma \hookrightarrow X$ with $i_m \circ \sigma = \sigma \circ i_m$.

Now let $d = 1$, $\vartheta = \gamma$ in Theorem 3.2 and let R be the corresponding residual subset of Σ_+. Then any $\omega \in R$ gives rise to an almost automorphic sequence $x_\omega \in \Sigma$ such that $M(\omega) = (\overline{O_\sigma(x_\omega)}, \sigma)$ is not uniquely ergodic, exhibits a topological entropy arbitrarily close to $\log m$, and has $(\mathbb{T}^1, R_\vartheta)$ as its maximal almost periodic factor. Applying Theorem 5.1, we obtain an almost automorphic function $f = \Phi_1 i_m(x_\omega) \in C(\mathbb{R}, \mathbb{R})$ such that the system $(H(f), \theta) = \big(H(i_m(x_\omega)), \theta\big)$ is not uniquely ergodic, exhibits large topological entropy, and has the Kronecker flow $(R_{(t,\vartheta t)})_{t\in\mathbb{R}}$ as its maximal almost periodic factor. Again, f admits two (basic) frequencies $\{1, \vartheta\}$.

Example 5.3. (*Chaotic almost automorphic signals with infinitely many frequencies*) Within the setting of Example 5.2, we now apply Theorem 3.4 with $d = 1$, $p_1 = p$ and let R be the residual subset of Σ_+ as in that theorem. For each $\omega \in R$, the generated almost automorphic sequence $x_\omega \in \Sigma$ and its orbit closure $M(\omega) = \overline{O_\sigma(x_\omega)}$ have the same properties as described in Example 5.2, except for the fact that the maximal almost periodic factor of $M(\omega)$ is now the p-adic odometer $(\Delta_p, \oplus_1)$. Let $f := \Phi_1 i_m(x_\omega) \in C(\mathbb{R}, \mathbb{R})$ be the almost automorphic function constructed as in Theorem 5.1. Then $(H(f), \theta) = (\overline{O_\theta(f)}, \theta)$ is not uniquely ergodic with nearly maximal entropy and has the suspension flow $(\Delta_p^{\oplus_1}, \varphi^{\oplus_1})$ as its maximal almost periodic factor. The group $\Delta_p^{\oplus_1} \sim \mathbb{T}^1 \times \Delta_p$ is a solenoid which can be visualized by means of the map $\Psi : \Delta_p^{\oplus_1} \to (\mathbb{T}^1)^{\mathbb{N}_0}$ defined according to

$$\Psi : (a, x) = (a, \sum_{j=0}^{\infty} x_j p^j) \mapsto (e^{2\pi i \psi_n})_{n\in\mathbb{N}_0} \quad \text{with} \quad \psi_n := (a + \sum_{j=0}^{n-1} x_j p^j) p^{-n} \, ;$$

for $n = 0$ the empty sum is understood to equal zero. It is not difficult to see that Ψ maps $\Delta_p^{\oplus_1}$ isomorphically to the inverse limit space $\Gamma_p := \{(z_n) \in (\mathbb{T}^1)^{\mathbb{N}_0} : z_n = z_{n+1}^p \text{ for all } n\} \subset (\mathbb{T}^1)^{\mathbb{N}_0}$. Since the dual group of $\mathbb{T}^1 \times \Delta_p$ is isomorphic to $\mathbb{Z} \times \{k/p^n : k \in \mathbb{Z}, n \in \mathbb{N}\}$, f admits countably many (basic) frequencies $\big\{1/p^n : n \in \mathbb{N} \cup \{0\}\big\}$.

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